MATH-449 - Biostatistics
EPFL, Spring 2023
Problem Set 4

1. (Exercise 2.4 in ABG 2008) Let M be a discrete time martingale with respect to the filtration $\mathcal{F}_n$, for $n \in \{0, 1, 2, \cdots\}$, and suppose $M_0 = 0$. Prove that $M^2 - \langle M \rangle$ is a martingale with respect to the filtration $\mathcal{F}$, that is, that $E(M_n^2 - \langle M \rangle_n | \mathcal{F}_{n-1}) = M_{n-1}^2 - \langle M \rangle_{n-1}$.

2. Suppose we have n independent survival times $\{T_i\}_{i=1}^n$, where T_i corresponds to the time of death of individual i. Suppose we somehow could observe each individual from $t = 0$ up to his/her time of death.

   In the lectures you learned that a counting process $\{N(t)\}_{t \geq 0}$ is an increasing right-continuous integer-valued stochastic process such that $N(0) = 0$. Write down the counting process N_i^c (that "counts" the death of individual i) in terms of T_i.

   You have also learned about the *intensity process* λ of a counting process N with respect to a filtration $\mathcal{F}$. It is informally defined through the relationship $\lambda(t)dt = E[dN(t)|\mathcal{F}_t]$.

   In general, if the intensity $\lambda(t)$ of a counting process $N(t)$ with respect to $\mathcal{F}_t$ can be written on the form

   $$\lambda(t) = \alpha(t) \cdot Z(t),$$

   where α is an unknown deterministic function and Z is an $\mathcal{F}_t$-predictable[§] function that does not depend on α, $N(t)$ is said to satisfy the *multiplicative intensity model*[*].

3. (Exercise 1.10 in ABG 2008) Consider the scenario in Exercise 2, and let $\mathcal{F}_t^c$ be the filtration generated by $\{N_i^c(s), s \leq t, i = 1, \cdots, n\}$. In the lectures we have seen that the intensity of N_i^c with respect to $\mathcal{F}^c$ in this case is $\lambda_i^c(t) = E[dN_i^c(t)|\mathcal{F}_t^c] = \alpha_i(t)Z_i(t)$, where $\alpha_i(t)$ is the hazard function of individual i and $Z_i(t) = I(T_i \geq t)$. Consider the aggregated counting process $N^c(t) = \sum_{i=1}^n N_i^c(t)$.

   i) Let $\{\eta_i(t)\}_{i=1}^n$ be known, positive, continuous functions. Find the intensity process of N^c with respect to $\mathcal{F}_t^c$ when α_i take the following forms:
      a) $\alpha_i(t) = \alpha(t)$
      b) $\alpha_i(t) = \eta_i(t)\alpha(t)$
      c) $\alpha_i(t) = \alpha(t) + \eta_i(t)$

   ii) For which of the three cases in i) does N^c satisfy the multiplicative intensity model?

4. Let N be a nonhomogeneous Poisson process with deterministic intensity function $\alpha(t)$. Define $H(t) = \int_0^t \alpha(s)ds$. The following two points i)-ii) provide equivalent definitions of such a process:

   i)
      - $N(t) - N(s) \sim \text{Poisson}(H(t) - H(s))$ for $s < t$
      - $N(t) - N(s)$ is independent of $\mathcal{F}_s$ for $s < t$

   ii)

   $$P(N_{t+\delta} - N_t = 1|\mathcal{F}_t) = \alpha(t)\delta + o(\delta^2)$$
   $$P(N_{t+\delta} - N_t = 0|\mathcal{F}_t) = 1 - \alpha(t)\delta + o(\delta^2)$$

   as $\delta \longrightarrow 0^+$.

   Here, $\mathcal{F}$ is the filtration generated by N. The second condition in i) implies that $E[N(t) - N(s)|\mathcal{F}_s] = E[N(t) - N(s)]$.

[§]Recall that this holds when Z is left-continuous and adapted to $\mathcal{F}$, i.e. that all the information needed to know the value of Z at time t is contained in $\mathcal{F}_t$.

[*]We will later derive estimators for the unknown function α under the multiplicative intensity model.

a) Show that $M = N - H$ is a martingale with respect to $\mathcal{F}$.*

b) Show that the increments of M are uncorrelated, i.e. that, for $v \le u \le s \le t$, †

$$E[(M(t) - M(s))(M(u) - M(v))] = 0.$$

Suppose that N is only recorded up to the deterministic time X, and define $N^\star(t) = N(\min\{t, X\})$. Thus, $N^\star$ is the process N censored at X.

c) Argue that $N^\star(t)$ is the observed number of jumps of N up to time t, and demonstrate that $N^\star$ satisfies the multiplicative intensity model.‡

d) Suppose now that X is a random variable. Verify that the conclusion in c) holds when $\{X \le t\} \in \mathcal{F}_t$ for each t, or equivalently, that $I(X \le \cdot)$ is adapted to $\mathcal{F}$.§

5. In this problem we will use the definition of the optional variation process $[\cdot]$ from the lecture notes. Thus, we will need to take limits $[G](t) = \lim_{n\to\infty} \sum_{k=1}^{n} \big(G(kt/n) - G((k-1)t/n)\big)^2$ (in probability) of processes G.

   Let $\{N(t) : t \in [0, \tau]\}$ be a counting process. Let λ be the intensity of N with respect to some filtration $\mathcal{F}$, so that $\Lambda(t) = \int_0^t \lambda(s)ds$ is the cumulative intensity, and $M = N - \Lambda$ is a martingale with respect to $\mathcal{F}$. Assume that $\int_0^\tau \lambda(s)^2 ds \le K$ for some constant K.

   a) Show that N has a finite number of jumps with probability 1. Hint: start by looking at $E[N(\tau)]$, use that M is a martingale and that $\int_0^\tau \lambda(s)^2 ds \le K$.¶

   b) Show that the optional variation process $[N]$ is equal to N (recall that there are no tied event times, so that $N(t) - N(t-) \le 1$ for all t).

6. Suppose $M = \{M_0, M_1, M_2, \dots\}$ is a discrete Martingale. Show that $Cov(M_m, M_n - M_m) = 0, \forall\, n > m$.

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*Hint: A Poisson distributed variable with parameter $\lambda > 0$ has mean λ.

†Note: this is true for any martingale M, not just the one from a).

‡Hint: start with definition ii). Alternatively, you may find it helpful to use $N^\star(t) = \int_0^t I(X \ge s)dN_s$.

§X is then called a *stopping time* with respect to $\mathcal{F}$. Heuristically, $\mathcal{F}_t$ contains enough information to determine whether X has occurred by t.

¶Hint: Use also the inequality $\big(\int_a^b f(s)ds\big)^2 \le (b-a)\int_a^b f(s)^2 ds$.