MATH-449 - Biostatistics
EPFL, Spring 2025
Problem Set 12

1. Show that the standard stochastic SIR model for the number of susceptibles ($X(t)$), and the number of infectives ($Y(t)$), introduced during the lecture, has the Markov property, if and only if, the (continuous) random variable I, denoting infectious period of infectives, is exponentially distributed.

2. Compute P_0^n, P_1^n and P_2^n numerically using the recursive formula given during the lecture (Equation 2.4 in the notes), assuming $n = 10$, $m = 1$, $\lambda = 2$ and that the infectious period I is:

   (a) exponentially distributed (the Markovian case) with mean 1 time unit.

   (b) $\Gamma(2, 2)$-distributed (i.e. with mean 1).

   (c) constant and equal to 1.

   Hint: You are welcome to calculate everything by hand, but I would advise using some programming language to derive these probabilities recursively.

3. Consider the Markovian version of the standard SIR epidemic (m fixed, n large). Without referring to the branching approximation, approximate the process of infectives $Y_n(t)$ during the initial stage of the epidemic with a suitable simple birth and death process. What is the probability of extinction/explosion of this approximating process?

   Hint: $X_n(t) \approx n$ during the initial stage of the epidemic.

   Birth and death processes are continuous-time Markov chains, with transition probabilities such that $P_{ij}(t) = 0$ if $|i - j| > 1$. In particular

$$
\begin{aligned}
P_{i,i+1}(dt) &= \lambda_i dt + o(dt),\\
P_{i,i-1}(dt) &= \mu_i dt + o(dt),\\
P_{j,j}(dt) &= 1 - (\lambda_j + \mu_j)dt + o(dt).
\end{aligned}
$$

4. **Challenging Exercise:**
   Suppose that $X = \{X(t); t \geq 0\}$ and $X' = \{X'(t); t \geq 0\}$ are two **birth and death processes** on the set of nonnegative integers. The process X has birth rates λ_i and death rates μ_i,

$$
\begin{aligned}
P(X(t + dt) - X(t) = +1 \mid X(t) = i) = \lambda_i dt + o(dt),\\
P(X(t + dt) - X(t) = -1 \mid X(t) = i) = \mu_i dt + o(dt),
\end{aligned}
$$

   $X(0) = m$, and likewise X' has birth rates λ_i', death rates μ_i' and initial value m'.

   Coupling is a mathematical technique via which we define highly dependent random elements, facilitating the comparison between random variables. Formally, given:

   - Two probability spaces: $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$
   - Random elements: $X : \Omega \to E$, and $X' : \Omega' \to E$
   - State space E (e.g. $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^{\mathbb{N}}$, $D[0, \infty)$, etc.),

   then we define the **coupling** of X and X' as:

   - A new probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$
   - A pair of random elements $(\widehat{X}, \widehat{X}') : \widehat{\Omega} \to E^2$

- Such that:

$$\widehat{X} \stackrel{d}{=} X \quad \text{and} \quad \widehat{X}' \stackrel{d}{=} X'$$

  (i.e. marginal distributions preserved)

We could use coupling to show that if $\lambda_i \leq \lambda_i'$ for all $i \geq 0$ and $\mu_i \geq \mu_i'$ for all $i \geq 1$, then $X(t)$ is stochastically smaller than $X'(t)$ for all t (provided also $m \leq m'$).

Define a bivariate process $(\hat{X}, \hat{X}')$ with initial value (m, m') and with the following intensity table:

from	**to**	**at rate**
(i,j)	$(i+1,j)$	λ_i
(i,j)	$(i,j+1)$	λ_j'
(i,j)	$(i-1,j)$	μ_i
(i,j)	$(i,j-1)$	μ_j'
(i,i)	$(i+1,i+1)$	λ_i
(i,i)	$(i,i+1)$	$\lambda_i' - \lambda_i$
(i,i)	$(i-1,i-1)$	μ_i'
(i,i)	$(i-1,i)$	$\mu_i - \mu_i'$

Check that the process $(\hat{X}, \hat{X}')$, is indeed a coupling of the birth and death processes X and X', i.e. that the marginal distributions coincide with the distributions of X and X', respectively.

Hint: Try to compute the marginal intensities of the joint Markov chain.