Exercises for Statistical analysis of network data – Sheet 6

1. Consider the network of the complete graph K_n on n vertices. Recall that the graph Laplacian is given by $L = \text{diag}(d_1, \ldots, d_n) - A = D - A$ and that the normalized graph Laplacian is $\mathcal{L} = D^{-1/2} L D^{-1/2}$.

   a) Calculate the degrees of this network by calculating $d = A\mathbf{1}$.

   b) Calculate the graph Laplacian, and the normalized Laplacian for this network.

   c) For $n = 5$, multiply the normalized Laplacian by $\mathbf{e}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

   d) The characteristic equation of a matrix $\{\mathcal{L}\}$ is given by

$$\|\mathcal{L} - \lambda \mathbf{I}\| = 0. \tag{1}$$

   Solve this equation in λ for the complete graph on 3 nodes. Factorize the characteristic equation down to the level you are able to.

2. Consider a star on 4 nodes. The characteristic equation of a matrix $\mathcal{L}$ is given by

$$\|\mathcal{L} - \lambda \mathbf{I}\| = 0. \tag{2}$$

   Factorize the characteristic equation down to the level you are able to in λ.

3. Take as the adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

   a) Calculate the degrees of this network by calculating $d = A\mathbf{1}$.

   b) Calculate the graph Laplacian, and the normalized Laplacian for this network.

   c) The characteristic equation of a matrix $\{\mathcal{L}\}$ is given by

$$\|\mathcal{L} - \lambda \mathbf{I}\| = 0. \tag{3}$$

   Solve this equation in λ. Factorize the characteristic equation down to the level you are able to.

   d) For the values of λ determine the vectors of

$$\mathcal{L}\mathbf{e} = \lambda \mathbf{e}.$$

4. Take as the adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

   a) Calculate the degrees of this network by calculating $d = A\mathbf{1}$.

   b) Calculate the graph Laplacian, and the normalized Laplacian for this network.

c) The characteristic equation of a matrix $\{\mathcal{L}\}$ is given by

$$\|\mathcal{L} - \lambda \mathbf{I}\| = 0. \tag{4}$$

Solve this equation in λ. Factorize the characteristic equation down to the level you are able to.

d) For the values of λ determine the vectors of

$$\mathcal{L}\mathbf{e} = \lambda \mathbf{e}.$$

e) Describe how to implement spectral clustering.