Statistical Theory MATH-442

Spring 2025 Ho Yun & Yoav Zemel

Exercise sheet 6

Exercise 1 Let $X_1, \dots, X_n$ be a sample from the uniform distribution on $(0, \theta)$ where $\theta > 0$ is unknown. Recall that the maximum likelihood estimator of θ is $X_{(n)} := \max(X_1, \dots, X_n)$. Consider estimators of θ of the form $\widehat{\theta}_b = bX_{(n)}$, $b > 0$. Find the estimator of this form that has the smallest risk $R(\theta, \widehat{\theta}_b) = \mathrm{E}_\theta \mathcal{L}(\theta, \widehat{\theta}_b)$ for all values of $\theta > 0$ (if such an estimator exists). Do this for the squared error loss function $\mathcal{L}_2(\theta, a) = (a - \theta)^2$ and for the absolute error loss function $\mathcal{L}_1(\theta, a) = |a - \theta|$.

Exercise 2 Let $X_1, X_2, \dots, X_n$ be a sample from a $N(\mu, \sigma^2)$ distribution, where $n > 1$ and both μ and σ^2 are unknown. The MLE of σ^2 can be shown to be

$$S_n^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2.$$

(you do not need to prove this, but if you never this result, you should prove it).

(a) Show that S_n^2 is inadmissible for σ^2 for the squared error loss function.

(b) Find the risk of the minimax estimator of σ^2 in the class of estimators of the form $a_n \sum_{i=1}^n (X_i - \overline{X})^2$ when $\sigma^2 \in (0, M]$, $M < \infty$.

(c) Why do we consider $\sigma^2 \in (0, M]$ in the previous point? What fails if we consider $\sigma^2 \in (0, \infty)$?

Exercise 3

1. Show that a unique minimax rule is admissible.
2. Show that a Bayes rule with constant risk is minimax.
3. Show that an admissible estimator with constant risk is minimax.

Exercise 4 Assume that $\Theta = \{\theta_1, \dots, \theta_t\}$ is a finite parameter space and the space of decision rules $\mathcal{D}$ is such that it includes all randomised rules. Define the risk set to be a subset S of $\mathbb{R}^t$ of the form $S = \{(R(\theta_1, d), \dots, R(\theta_t, d)) : d \in \mathcal{D}\}$.

Show that S is a convex set.

Exercise 5 Consider a parameter space with two values $\Theta = \{\theta_1, \theta_2\}$. In each plot in Figure 1, coordinates are $r_1 = R(\theta_1, d)$, $r_2 = R(\theta_2, d)$, the dots and/or thick curves correspond to the values (r_1, r_2) of risk of some non-randomised decision rules, the filled ares are risk sets consisting of points corresponding to all non-randomised and randomised decisions (all convex combinations of the points corresponding to non-randomised decisions). For each risk set:

(a) Draw the set of admissible decisions.

(b) Draw curves corresponding to the decision rules with the same value of the maximal risk $\max(R(\theta_1, d), R(\theta_2, d))$ (i.e., for various values of c, draw "iso-max-risk" curves satisfying $\max(r_1, r_2) = c$).

(c) Use these curves to find the minimax decision(s). Discuss whether it is (they are) unique, randomised, admissible.

(d) Suppose we have prior probabilities $\pi_1 = \pi(\theta_1) \geq 0$, $\pi_2 = \pi(\theta_2) \geq 0$, $\pi_1 + \pi_2 = 1$. Draw curves corresponding to the decisions with the same value of the Bayes risk $\pi_1 R(\theta_1, d) + \pi_2 R(\theta_2, d)$ (i.e., for various values of c, draw "iso-Bayes-risk" curves satisfying $\pi_1 r_1 + \pi_2 r_2 = c$). Do this and the next step for various prior probabilities, for example for $\pi_1 = 0.5, 0.25, 0.75, 0, 1$.

(e) Use these curves to find the Bayes decision(s). Discuss if it is (they are) unique, randomised, admissible.

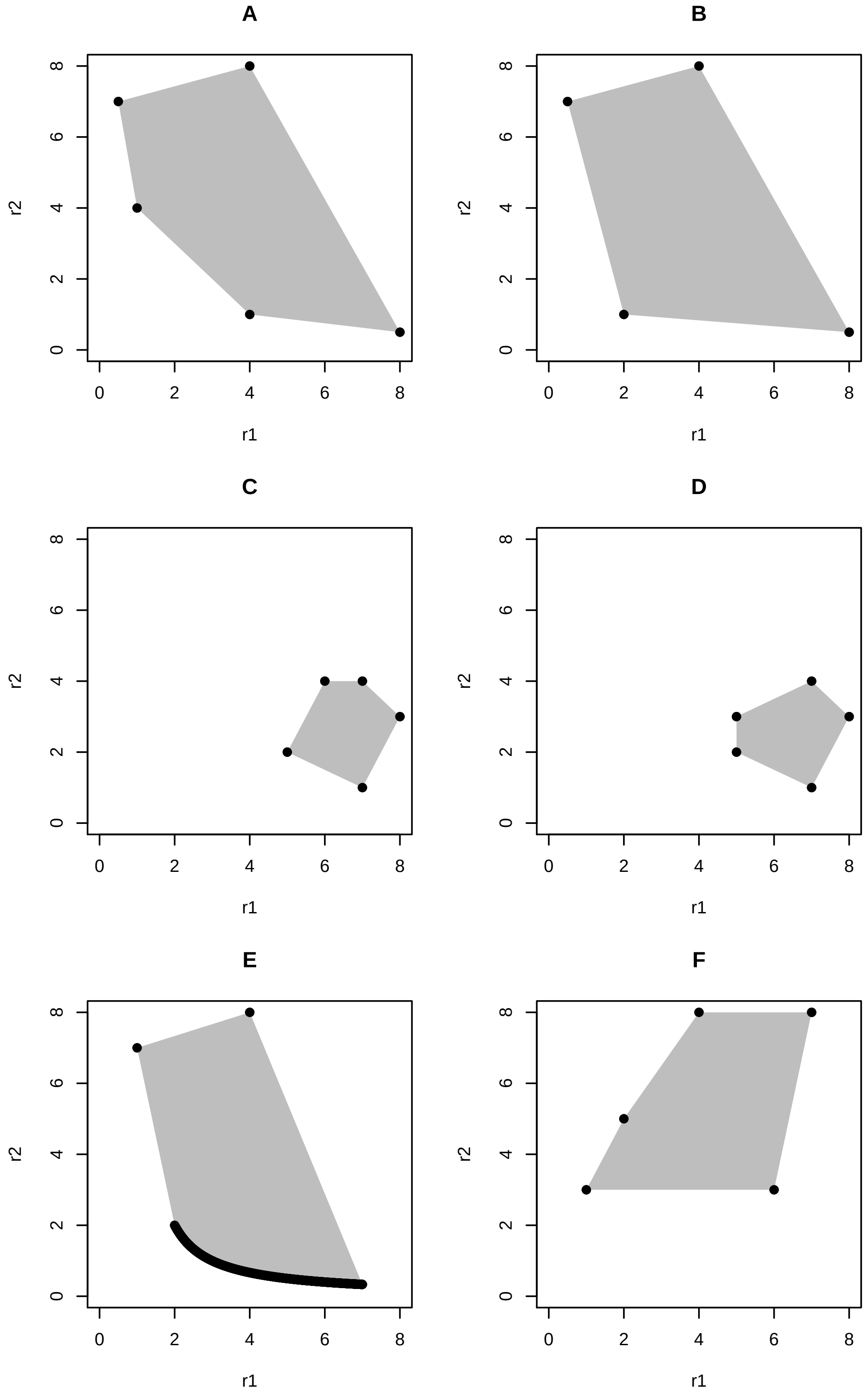

Figure 1: Risk values for non-randomised (black) and randomised (grey) decision rules