
Exercise sheet 5

In these exercise set, you may use without proof that $\Gamma'(p)/\Gamma(p)$ is strictly increasing in p with limit $-\infty$ as $p \searrow 0$ and ∞ as $p \to \infty$.

Exercise 1

1. Suppose that $X_n \xrightarrow{d} X$ on $\mathbb{R}^k$. Show that (X_n) is bounded in probability.
2. Show that if $X_n \xrightarrow{d} 0$ and Y_n is bounded in probability, then $X_n Y_n \xrightarrow{d} 0$. Here we assume that $X_n Y_n$ makes sense, so that either one of them is scalar, or X_n is a matrix and Y_n and appropriate vector, or X_n and Y_n are vectors of the same dimension and we take an inner product, etc.

Exercise 2 In the one-dimensional case, examine the conditions under which an estimator T attains the Cramér–Rao lower bound.

Exercise 3 Let $X_1, \dots, X_n$ be a sample from the logistic distribution with density

$$f(x;\mu) = \tfrac{\exp\{-(x-\mu)\}}{(1+\exp\{-(x-\mu)\})^2}, \qquad x \in \mathbb{R}, \quad \mu \in \mathbb{R}.$$

(a) Find the maximum likelihood estimator of μ (and verify that the solution actually maximises the likelihood), compute the Fisher information and find the asymptotic distribution of the estimator as $n \to \infty$.

(b) Find an estimator of μ that has an explicit expression and the same asymptotic properties as the maximum likelihood estimator.

Exercise 4 Let $X_1, \dots, X_n$ be a sample from the Gamma distribution with known a (rate) and unknown p (shape) (the density is $f(x;a,p) = \frac{a^p}{\Gamma(p)} x^{p-1} e^{-ax} 1_{(0,\infty)}(x)$). Find an estimator $\tilde{p}$ of p and use it to construct a one-step estimator p^*. Find the asymptotic distribution of $\sqrt{n}(p^* - p)$.

Exercise 5 Let $X_1, \dots, X_n$ be a sample from a Weibull distribution with density

$$f(x;a,p) = apx^{p-1} \exp\{-ax^p\} 1_{(0,\infty)}(x),$$

where the parameter a is known to be 1 while the value of $p > 0$ is unknown. Suppose however that we have falsely assumed that the distribution is $\Gamma(1,p)$ (density $\frac{1}{\Gamma(p)} x^{p-1} e^{-x} 1_{(0,\infty)}(x)$) with parameter $p > 0$ estimated by the maximum likelihood estimator $\widehat{p}_n^{MLE}$. What does $\widehat{p}_n^{MLE}$ estimate? Find its asymptotic distribution.
Hint: you may use without proof that $\int_0^\infty e^{-y} \log y dy = -\gamma \approx .577$ (the Euler–Mascheroni constant).

Exercise 6 Let $X_1, \dots, X_n$ be a sample from the Weibull distribution with the density $f(x;\lambda,p) = (\lambda p)\,(\lambda x)^{p-1}\, e^{-(\lambda x)^p}$ for $x > 0$, where $\lambda > 0$ and $p > 0$ are unknown parameters. Suppose that we thought that the distribution was exponential with the density $g(x;\lambda) = \lambda e^{-\lambda x} 1(x > 0)$ and calculated the maximum likelihood estimator $\widehat{\lambda}_n^{MLE}$. What does $\widehat{\lambda}_n^{MLE}$ estimate and what are its asymptotic properties? Compare with the maximum likelihood estimator of λ computed under the correct model specification when the parameter p of the Weibull distribution is known.