Statistical Theory MATH-442
Spring 2025 Ho Yun & Yoav Zemel

Exercise sheet 4

Exercise 1 Consider independent Gaussian random variables $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ with $X_j, Y_j \sim N(\mu_j, \sigma^2)$ with $\mu_1, \ldots, \mu_j, \sigma^2$ unknown. Find the maximum likelihood estimator. Is it consistent?

Exercise 2 Consider the geometric distribution with parameter $p \in (0,1]$ ($P(X = x) = (1-p)^x p$, $x = 0, 1, \ldots$ and $\mathbb{E}_p X = (1-p)/p$).

Find the maximum likelihood estimator $\widehat{p}_n^{MLE}$ and show that it is biased, that is $\mathbb{E}_p \widehat{p}_n^{MLE} \neq p$. *Hint:* Subtract and add $\mathbb{E}_p \overline{X}_n = (1-p)/p$ in the denominator and use an inequality based on two terms of a geometric series.

Exercise 3 Let $X_1, \ldots, X_n$ be a sample from the distribution with density

$$f(x; \theta, p) = (1-p)1_{(-1,0)}(x) + p\theta^{-1}1_{(0,\theta)}(x)$$

(a mixture of $U(-1,0)$ and $U(0,\theta)$), where $p \in [0,1]$ and $\theta > 0$. Find the maximum likelihood estimators of the parameters p and θ.

Exercise 4 Find the asymptotic covariance of the maximum likelihood estimator from question 5 of last week.

Exercise 5 Consider the rescaled Beta$(1, \alpha + 1)$ distribution with known $\alpha > -1$ and density

$$f(x; \theta) = (\alpha + 1)(\theta - x)^\alpha \theta^{-\alpha-1} 1(x \in [0, \theta]),$$

where $\theta > 0$ is unknown. Let $X_1, \ldots, X_n \overset{iid}{\sim} f(x; \theta_0)$ for some $\theta_0 > 0$.

(a) Investigate the regularity conditions $\mathbb{E}_\theta S_n(\theta) = 0$ and $I_n(\theta) = J_n(\theta)$ as a function of $\alpha > -1$.

(b) It is not easy to show that $\widehat{\theta}_n^{MLE}$ is consistent. But show that $T_n \leq \widehat{\theta}_n^{MLE} \leq (\alpha + 1)T_n$, where T_n is a consistent estimator.