Statistical Theory MATH-442
Spring 2025 Ho Yun & Yoav Zemel

Exercise sheet 1

You may wish to refer to the probability recap document on Moodle about definitions of some distributions, if needed.

Exercise 1 If X is exponentially distributed with intensity λ, what is the distribution of $Y = \lfloor X \rfloor$, the largest integer less than or equal to X?

Exercise 2 Suppose that $S \sim \text{Exp}(\lambda)$, $C \sim \text{Exp}(\gamma)$ are independent. Define $T = \min(S, C)$ and $D = \mathbf{1}[T = S]$. Find the joint distribution of T and D and their marginal distributions. Are T and D independent?

Exercise 3 Consider a random vector $(X, Z)^T$. Let the marginal distribution of Z be exponential with parameter γ, i.e., with density $f_Z(z) = \gamma e^{-\gamma z} 1_{(0,\infty)}(z)$. Suppose that the conditional distribution of X given $Z = z$ is Poisson with parameter λz, i.e.,

$$P(X = x | Z = z) = \tfrac{(\lambda z)^x}{x!} e^{-\lambda z}, \quad x = 0, 1, \ldots, z > 0.$$

1. Find the joint density $f_{X,Z}(x, z)$.
2. Compute the marginal (unconditional) distribution of X. Which known distribution is it?
3. Find $E[X|Z]$.
4. Find $E[X]$.
5. Find $\text{var}(X|Z)$.
6. Compute $\text{var}(X)$.
7. Compute the conditional density $f_{Z|X}(z|x)$. Which known distribution is it?

Exercise 4 Show that for a nonnegative random variable X,

$$\mathbb{E}X = \int_0^\infty P(X > t)dt \in [0, \infty].$$

Try to prove it without assuming the existence of a density/mass function. *Hint: Use Fubini/Tonelli theorem.*

Remark. This result is true even if $\mathbb{E}[X]$ is finite, since we are applying the theorem over σ-finite measures: Lebegue meausure on $\mathbb{R}$ and the finite probability measure corresponding to X.

Exercise 5 Note: hint on the following page! This is a nice (in the lecturer's opinion!) geometrical result we shall need in the part of the course about classification. It can be interpreted as the triangle inequality for angles. For $d \geq 2$ and nonzero vectors $y, z \in \mathbb{R}^d$ define the angle between them by

$$\theta(y, z) = \cos^{-1} \tfrac{y^\top z}{\|y\|\|z\|} \in [0, \pi].$$

Let $v \in \mathbb{R}^d \setminus \{0\}$ be an additional vector such that $\theta(y, v) \leq \pi/2$ and $\theta(z, v) \leq \pi/2$. Show that

$$\theta(y, z) \leq \theta(y, v) + \theta(v, z).$$

Hint: it might be easier to restrict (with justification) to the case where v *is the first unit vector, and all vectors have norm one. You may also recall the formula* $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, *and use that cos is decreasing on* $[0, \pi]$.

Remark. This is also true, but not really interesting, if $d = 1$.