Distribution and Interpolation Theory

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The questions are independent of each other in the sense that one can admit previous answers to treat the next questions. One need not solve all questions to get the maximal grade, and questions can be solved in any order. Good luck! Bonne chance ! Viel Erfolg!

Exercise 1 (Calderón's theorem)**.** In this exercise, we will prove Calderón's following result.

Theorem. *Let $d \geq 2$, and $\Omega \subset \mathbb{R}^d$ be a bounded open connected domain. Then, the vector space space spanned by products of* real-valued *harmonic functions is dense in $L^2(\Omega, \mathbb{R})$.*

Recall that a real-valued function $u \in C^\infty(\overline{\Omega}, \mathbb{R})$ is said harmonic if $\Delta u = 0$, where

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$$

is the Laplacian operator.

We let $\mathscr{H} = C^\infty(\overline{\Omega}, \mathbb{R}) \cap \{u : \Delta u = 0 \quad \text{in } \Omega\}$, $\Pi = \text{Span}\,\{u \cdot v : (u, v) \in \mathscr{H} \times \mathscr{H}\}$.

(1) Let

$$A = \mathbb{C}^d \cap \left\{ \zeta : \zeta \cdot \zeta = \sum_{j=1}^{d} \zeta_j^2 = 0 \right\}.$$

Show that for all $\zeta \in A$, the function $u : \Omega \to \mathbb{C}, u(x) = e^{i\, x \cdot \zeta}$ is harmonic.

(2) Fix some $f \in L^2(\Omega, \mathbb{R})$, and assume that

$$\int_{\Omega} f\, w\, dx = 0 \quad \text{for all } \; w \in \Pi.$$

Using well-chosen functions defined in (1), show that $\mathscr{F}(f\,\mathbf{1}_\Omega) = 0$, where $\mathbf{1}_\Omega$ is the indicator function of Ω, and $\mathscr{F}$ is the Fourier transform. What can you deduce about f?

(3) Conclude the proof of the theorem.

Exercise 2 (Basic estimates for the Schrödinger equation)**.** (1) Let $d \geq 1$, and $\alpha > 0$ be a fixed real number. Using the formula

$$\mathscr{F}\left(\mathbb{R}^d \to \mathbb{R}, x \mapsto e^{-\alpha x^2}\right)(\xi) = \left(\frac{\pi}{\alpha}\right)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4\alpha}}$$

for all $\xi \in \mathbb{R}^n$, show that the function $g_\alpha : \mathbb{R}^n \to \mathbb{C}, x \mapsto e^{-i\alpha|x|^2}$ is a tempered distribution (an element of $\mathscr{S}'(\mathbb{R}^d)$), and show that

$$\widehat{g}_\alpha(\xi) = \left(\frac{\pi}{i\,\alpha}\right)^{\frac{d}{2}} e^{i\frac{|\xi|^2}{4\alpha}}.$$

(2) Let $f \in \mathscr{S}(\mathbb{R}^d)$, and consider the following partial differential equation:

$$\begin{cases} i\,\dfrac{\partial}{\partial t}u + \Delta u = 0 & \text{in } \; (0,\infty) \times \mathbb{R}^d \\ \qquad u(0,x) = f(x) & \end{cases}, \tag{1}$$

where we recall that the Laplacian Δ is defined by

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2},$$

and $u \in C^1((0,\infty), \mathscr{S}'(\mathbb{R}^d))$. Using the previous question, show that the following representation formula holds for u:

$$u(t,x) = \frac{1}{(2\pi\, i\, t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} f(y) dy \quad \text{for all } \; (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d.$$

(3) Show that $u \in C^\infty((0,\infty)\times\mathbb{R}^d)$, and that for all $t>0$,

$$\|u(t\,\cdot)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} |u(t,x)| \leq \frac{1}{(2\pi t)^{\frac{d}{2}}} \|f\|_{\mathrm{L}^1(\mathbb{R}^d)},$$

and that for all $t>0$, we have

$$\|u(t,\,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d)} = \|u(0,\,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d)} = \|f\|_{\mathrm{L}^2(\mathbb{R}^d)}.$$

More generally, estimate for all $s\in\mathbb{R}$ the following norm $\|u(t,\,\cdot)\|_{\mathrm{H}^s(\mathbb{R}^d)}$ $(t>0)$ in terms of $\|f\|_{\mathrm{H}^s(\mathbb{R}^d)}$.

(4) Let $2\leq p\leq\infty$. By a scaling argument, show that provided that the inequality

$$\|u(t,\,\cdot)\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq \frac{C}{t^\alpha}\|f\|_{\mathrm{L}^{p'}(\mathbb{R}^d)}$$

holds for all $f\in\mathscr{S}(\mathbb{R}^d)$ and all solution u of (1) (with initial data f), then $\alpha = d\left(\frac{1}{2}-\frac{1}{p}\right)$. One may introduce the following function $u_\lambda(t,x) = u(\lambda^2\,t,\lambda\,x)$ for $\lambda>0$ in the proof.

This estimate is one of the basic ingredients of the proof of the space-time Strichartz estimates*:*

$$\|u\|_{\mathrm{L}^p(\mathbb{R}_+,\mathrm{L}^q(\mathbb{R}^d))} \leq C(p,q,d)\,\|f\|_{\mathrm{L}^2(\mathbb{R}^d)}$$

for all $\frac{2}{p}+\frac{d}{q}=\frac{d}{2}$, $(p,q)\neq(2,\infty)$, $p\geq 2$.

Exercise 3 (Elliptic Regularity)**.** This exercise aims at generalising the *elliptic estimates* on the harmonic functions. Namely, $\Delta u = f\in C^\infty(\Omega)$ implies that $u\in C^\infty(\Omega)$. Here, we will prove the main step that shows that weak solution of elliptic partial differential equation are continuous.

Let $\Omega\subset\mathbb{R}^d$ be a connected open subset. Recall that for all $1\leq p,q\leq\infty$, for all $u\in W^{1,p}(\Omega,\mathbb{R}^d)$ and $\varphi\in W_0^{1,q}(\Omega)$ such that $u\nabla\varphi\in L^1(\Omega)$ and $\varphi\nabla u\in L^1(\Omega)$, we have for all $1\leq j\leq d$

$$\int_\Omega u\,\partial_{x_j}\varphi\,dx = -\int_\Omega \left(\partial_{x_j}u\right)\varphi\,dx.$$

Restricting to $\Omega = B(0,1)$ from now on, we let $A=(a_{i,j})_{1\leq j\leq d}\in L^\infty(B(0,1),\mathbb{R}^d)$ be a space-dependent *uniformly elliptic* matrix, *i.e.* there exists $0<\Lambda<\infty$ such that

$$\Lambda^{-1}|\xi|^2 \leq \xi^t A(x)\xi = \sum_{i,j=1}^d a_{i,j}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \forall x\in B(0,1),\forall\xi\in\mathbb{R}^d. \tag{1}$$

We now let $u\in W^{1,2}(B(0,1))$ a weak solution (in $\mathscr{D}'(B(0,1))$) of the linear partial differential equation

$$\operatorname{div}\left(A(x)\nabla u\right) = 0. \tag{2}$$

Important note: all inequalities can be proven with "worse" constants than stated, provided that they are universal constants (only depend on the ambient dimension and the parameter Λ*), unless stated otherwise.*

(1) For $d=1$, show directly that $u\in C^0(B(0,1))$.

(2) Show that for all $\varphi\in\mathscr{D}(B(0,1)) = C_c^\infty(B(0,1))$, the following identity holds

$$\int_{B(0,1)} A(x)\nabla u\cdot\nabla\varphi\,dx = 0.$$

Notice that for all $u,v\in H^1(B(0,1))$, we have

$$\int_{B(0,1)} A(x)\nabla u\cdot\nabla v\,dx = \sum_{i,j=1}^d \int_{B(0,1)} a_{i,j}(x)\partial_{x_i}u\,\partial_{x_j}v\,dx.$$

(3) (Caccioppoli inequality) Let $u \in W^{1,2}(B(0,1))$. Show that for all $\eta \in \mathscr{D}(B(0,1))$, and $c \in \mathbb{R}$, we have

$$\int_{B(0,1)} |\nabla u|^2 \eta^2 dx \leq \Lambda^2 \int_{B(0,1)} |u-c|^2 |\nabla \eta|^2 dx.$$

One can use the following test function $\varphi = (u-c)\eta^2$.

(4) By choosing an appropriate test function η, show that for all $0 < r_1 < r_2 < 1$, we have for some universal constant $0 < C < \infty$,

$$\int_{B(0,r_1)} |\nabla u|^2 dx \leq \frac{C}{(r_2 - r_1)^2} \int_{B(0,r_2)\setminus \overline{B}(0,r_1)} |u-c|^2 dx \tag{3}$$

This is the *Caccioppoli inequality.*

(5) Using the Poincaré-Wirtinger inequality and a scaling argument, show that for all for all $0 < r < \infty$ if $A(r) = B(0,2r) \setminus \overline{B}(0,r)$ and $u \in W^{1,2}(B(0,r))$, we have

$$\int_{A(r)} |u - u_r|^2 dx \leq C r^2 \int_{A(r)} |\nabla u|^2 dx,$$

where $u_r = \fint_{A(r)} u\, dx = \dfrac{1}{c_d\, r^d} \displaystyle\int_{A(r)} u\, dx$ (for some universal constant c_d) is the mean of u on $A(r)$, and $0 < C < \infty$ is a universal constant independent of r.

(6) Deduce from the previous questions that there exists a universal contant $0 < \theta < 1$ such that for all $0 < r < \dfrac{1}{2}$, we have

$$\int_{B(0,r)} |\nabla u|^2 dx \leq \theta \int_{B(0,2r)} |\nabla u|^2 dx.$$

By induction, deduce that there exists $0 < \alpha < 1$ and $0 < C < \infty$ such that for all $0 < r < \dfrac{1}{2}$, we have

$$\int_{B(0,r)} |\nabla u|^2 \leq C r^\alpha \int_{B(0,1)} |\nabla u|^2 dx.$$

Hint: use a dyadic argument.

(7) With the help of another optimal Poincaré-Wirtinger inequality, deduce that there exists a universal constant $0 < C < \infty$ such that for all $0 < r < \dfrac{1}{2}$,

$$\int_{B(0,r)} |u - \widetilde{u}_r|^2 dx \leq C r^{2+\alpha} \int_{B(0,1)} |\nabla u|^2 dx,$$

where $\widetilde{u}_r = \fint_{B(0,r)} u\, dx = \dfrac{1}{c_d' r^d} \displaystyle\int_{B(0,r)} u\, dx$ is the mean of u on $B(0,r)$.

(8*) In the special case $d = 2$, show with the help of Lebesgue differentiation theorem and a translation argument that $u \in C^{0,\beta}(B(0,\frac{1}{4}))$ for some $0 < \beta < 1$. Recall that the Lebesgue differentiation theorem shows that for all $u \in L^1_{\mathrm{loc}}(B(0,1))$, and for all $x \in B(0,1)$, we have[†]

$$\fint_{B(x,r)} u(y)\, dy = \frac{1}{\pi r^2} \int_{B(x,r)} u(y)\, dy \underset{r \to 0}{\longrightarrow} u(x).$$

[†]Notice that the theorem is stated on $\mathbb{R}^2$.

Furthermore, recall that the space of β-Hölder functions is defined by

$$C^{0,\beta}(\Omega) = C^0(\Omega) \cap \left\{ u : \|u\|_{C^{0,\beta}(\Omega)} = \|u\|_{\mathrm{L}^\infty(\Omega)} + \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\beta} < \infty \right\}$$

and equipped with the $\|u\|_{C^{0,\beta}(\Omega)}$ norm defined above. On may first prove the following estimate

$$|u_{x_0,r} - u_{x_0,s}| \leq C\frac{r^{1+\frac{\alpha}{2}}}{s}\int_{B(0,1)} |\nabla u|^2 dx$$

for all $x_0 \in B(0, \frac{1}{2})$, and $0 < s < r < \frac{1}{2}$, where $u_{x_0,r} = \fint_{B(x_0,r)} u\, dx = \frac{1}{\pi r^2}\int_{B(x_0,r)} u\, dx$.

(9) Assume from now on that $d \geq 3$. We say that $u \in H^1(B(0,1))$ is a weak sub-solution if for all $\varphi \in \mathscr{D}(B(0,1))$, we have

$$\int_\Omega A(x)\nabla u \cdot \nabla\varphi\, dx \leq 0, \tag{4}$$

or in other words

$$\sum_{i,j=1}^{d}\int_\Omega a_{i,j}(x)\partial_{x_i} u\, \partial_{x_j} v\, dx \leq 0.$$

Using the chain rule and the product formula for Sobolev functions, show that for all convex, increasing function $\Phi \in C^2(\mathbb{R}, \mathbb{R}_+)$ such that $\Phi''(t) = 0$ for all $|t| \geq R$ (for some fixed $R > 0$), then for any sub-solution $u \in H^1(B(0,1))$, the function $\Phi(u) \in H^1(B(0,1))$ is a sub-solution.

From now on, we assume that the previous result holds for any increasing, convex function $\Phi : \mathbb{R} \to \mathbb{R}_+$ (not necessarily C^2 or satisfying the extra hypothesis on the second derivative).

(10) (Moser iterations) Fix some $0 < 2r < 1$, and for all $j \in \mathbb{N}$, let $B_j = B(0, r + 2^{-j}r)$, so that $B_{j+1} \subset B_j \subset B(0, 2r)$ and

$$B_\infty = \bigcap_{j\in\mathbb{N}} B_j = B(0,r).$$

Show that for all $1 \leq \lambda \leq \dfrac{d}{d-2}$, there exists $0 < \gamma < \infty$ such that for all $j \in \mathbb{N}$, for all $v \in H^1(B_{j+1})$, we have

$$\left\||v|^\lambda\right\|^2_{\mathrm{L}^2(B_{j+1})} \leq \gamma\,\|\nabla v\|^{2\lambda}_{\mathrm{L}^2(B_{j+1})} + \gamma\,\|v\|^{2\lambda}_{\mathrm{L}^2(B_{j+1})}.$$

(11) Fix $1 < \lambda \leq \frac{d}{d-2}$, and let $u \in H^1(B_j)$ a *positive* weak sub-solution. With the help of the Caccioppoli inequality (explain why it holds true for a sub-solution), show that $u^\lambda \in H^1(B_{j+1})$, that u^λ is a positive weak sub-solution, and show the inequality

$$\left\|u^\lambda\right\|^2_{\mathrm{L}^2(B_{j+1})} \leq C(2^{2\lambda j}+1)\,\|u\|^{2\lambda}_{\mathrm{L}^2(B_j)},$$

where C only depends on $d, \Lambda, r > 0$.

(12) Let $u \in H^1(B(0,1))$ be a positive weak sub-solution. Define for all $j \in \mathbb{N}$ the function $u_j = u^{\lambda^j}$. Explain why

$$\|u\|_{\mathrm{L}^\infty(B(0,r))} = \limsup_{j\to\infty}\|u\|_{\mathrm{L}^{2\lambda^j}(B_j)},$$

and prove the following inequality

$$\|u\|_{\mathrm{L}^\infty(B(0,r))} \leq C(d,\Lambda,r)\,\|u\|_{\mathrm{L}^2(B(0,2r))}.$$

Solution

Exercise 1 (Calderón's theorem)**.** In this exercise, we will prove Calderón's following result.

Theorem. *Let $d \geq 2$, and $\Omega \subset \mathbb{R}^d$ be a bounded open connected domain. Then, the vector space space spanned by products of* real-valued *harmonic functions is dense in $L^2(\Omega, \mathbb{R})$.*

Recall that a real-valued function $u \in C^\infty(\overline{\Omega}, \mathbb{R})$ is said harmonic if $\Delta u = 0$, where

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$$

is the Laplacian operator.

We let $\mathscr{H} = C^\infty(\overline{\Omega}, \mathbb{R}) \cap \{u : \Delta u = 0 \quad \text{in } \Omega\}$, $\Pi = \mathrm{Span}\,\{u \cdot v : (u, v) \in \mathscr{H} \times \mathscr{H}\}$.

(1) Let

$$A = \mathbb{C}^d \cap \left\{ \zeta : \zeta \cdot \zeta = \sum_{j=1}^{d} \zeta_j^2 = 0 \right\}.$$

Show that for all $\zeta \in A$, the function $u : \Omega \to \mathbb{C}, u(x) = e^{i\,x \cdot \zeta}$ is harmonic.

(2) Fix some $f \in L^2(\Omega, \mathbb{R})$, and assume that

$$\int_\Omega f\, w\, dx = 0 \quad \text{for all } w \in \Pi.$$

Using well-chosen functions defined in (1), show that $\mathscr{F}(f\,\mathbf{1}_\Omega) = 0$, where $\mathbf{1}_\Omega$ is the indicator function of Ω, and $\mathscr{F}$ is the Fourier transform. What can you deduce about f?

(3) Conclude the proof of the theorem [thanks to Hahn-Banach theorem].

Proof.

(1) If $u(x) = e^{i\,x \cdot \zeta}$, we compute for all $1 \leq j \leq d$

$$\partial_{x_j} u = i\,\zeta_j\, u$$
$$\partial_{x_j}^2 u = -\zeta_j^2\, u.$$

Therefore, we get

$$\Delta u = \sum_{j=1}^{d} \partial_{x_j}^2 u = -\sum_{j=1}^{d} \zeta_j^2\, u = -(\zeta \cdot \zeta) u = 0,$$

which shows that u is harmonic.

(2) Since the real-part of harmonic functions is harmonic, for all $\zeta \in A$, we have

$$\mathrm{Re}\,(e^{i\,x\,\zeta})\mathrm{Re}\,(e^{i\,x\,\overline{\zeta}}), \mathrm{Re}\,(e^{i\,x\,\zeta})\mathrm{Im}\,(e^{i\,x\,\overline{\zeta}}), \mathrm{Im}\,(e^{i\,x\,\zeta})\mathrm{Re}\,(e^{i\,x\,\overline{\zeta}}), \mathrm{Im}\,(e^{i\,x\,\zeta})\mathrm{Im}\,(e^{i\,x\,\overline{\zeta}}) \in \Pi,$$

so the hypothesis shows that for all $\zeta \in A$, we have

$$\int_\Omega f(x)\, e^{2\,i\,x \cdot \mathrm{Re}\,(\zeta)}\, dx = 0.$$

Notice that for all $\xi \in \mathbb{R}^d$, there exists $\eta \in \mathbb{R}^d$ such that $\xi + i\,\eta \in A$. Indeed, it suffices to take η such that

$$|\eta|^2 = |\xi|^2, \quad \text{and } \langle \xi, \eta \rangle = \sum_{j=1}^{d} \xi_j\, \eta_j = 0.$$

One can simply take $\zeta \in \xi^{\perp} \simeq \mathbb{R}^{d-1}$ of norm equal to ξ which is non-empty since $d \geq 2$. Finally, taking $\zeta = -\frac{1}{2}(\xi + i\,\eta)$, we deduce that

$$\int_{\Omega} f(x) e^{-i\,x\cdot\xi} dx = 0.$$

Since this identity is valid for all $\xi \in \mathbb{R}^d$, we deduce that $\mathscr{F}(\mathbf{1}_{\Omega} f) = 0$, which shows by uniqueness of Fourier transform that $f = 0$.

(3) By Hahn-Banach theorem, if Π was not dense in $L^2(\Omega, \mathbb{R})$, there would exist a linear form $L : L^2(\Omega, \mathbb{R}) \to \mathbb{R}$ such that $L(w) = 0$ for all $w \in \Pi$. By Riesz-Fréchet representation theorem, since $L(w) = \int_{\Omega} f\, w\, dx$ for some $f \in L^2(\Omega, \mathbb{R})$, the previous question shows that $f = 0$, which is a contradiction. □

Exercise 2 (Basic estimates for the Schrödinger equation)**.** (1) Let $d \geq 1$, and $\alpha > 0$ be a fixed real number. Using the formula

$$\mathscr{F}\left(\mathbb{R}^d \to \mathbb{R}, x \mapsto e^{-\alpha x^2}\right)(\xi) = \left(\frac{\pi}{\alpha}\right)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4\alpha}}$$

for all $\xi \in \mathbb{R}^n$, show that the function $g_{\alpha} : \mathbb{R}^n \to \mathbb{C}, x \mapsto e^{-i\alpha|x|^2}$ is a tempered distribution (an element of $\mathscr{D}'(\mathbb{R}^d)$), and show that

$$\widehat{g}_{\alpha}(\xi) = \left(\frac{\pi}{i\,\alpha}\right)^{\frac{d}{2}} e^{i\frac{|\xi|^2}{4\alpha}}.$$

(2) Let $f \in \mathscr{S}(\mathbb{R}^d)$, and consider the following partial differential equation:

$$\begin{cases} i\,\dfrac{\partial}{\partial t} u + \Delta u = 0 & \text{in } (0,\infty) \times \mathbb{R}^d \\ \qquad u(0,x) = f(x) \end{cases}, \tag{5}$$

where we recall that the Laplacian Δ is defined by

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2},$$

and $u \in C^1((0,\infty), \mathscr{S}'(\mathbb{R}^d))$. Using the previous question, show that the following representation formula holds for u:

$$u(t,x) = \frac{1}{(2\pi\,i\,t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} f(y) dy \quad \text{for all } (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d.$$

(3) Show that $u \in C^{\infty}((0,\infty) \times \mathbb{R}^d)$, and that for all $t > 0$,

$$\|u(t\,\cdot)\|_{\mathrm{L}^{\infty}(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} |u(t,x)| \leq \frac{1}{(2\pi t)^{\frac{d}{2}}} \|f\|_{\mathrm{L}^1(\mathbb{R}^d)},$$

and that for all $t > 0$, we have

$$\|u(t,\,\cdot\,)\|_{\mathrm{L}^2(\mathbb{R}^d)} = \|u(0,\,\cdot\,)\|_{\mathrm{L}^2(\mathbb{R}^d)} = \|f\|_{\mathrm{L}^2(\mathbb{R}^d)}.$$

More generally, estimate for all $s \in \mathbb{R}$ the following norm $\|u(t,\,\cdot\,)\|_{\mathrm{H}^s(\mathbb{R}^d)}$ $(t > 0)$ in terms of $\|f\|_{\mathrm{H}^s(\mathbb{R}^d)}$.

(4) Let $2 \leq p \leq \infty$. By a scaling argument, show that provided that the inequality

$$\|u(t,\,\cdot\,)\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq \frac{C}{t^{\alpha}} \|f\|_{\mathrm{L}^{p'}(\mathbb{R}^d)}$$

holds for all $f \in \mathscr{S}(\mathbb{R}^d)$ and all solution u of (1) (with initial data f), then $\alpha = d\left(\frac{1}{2} - \frac{1}{p}\right)$. One may introduce the following function $u_\lambda(t,x) = u(\lambda^2\, t, \lambda\, x)$ for $\lambda > 0$ in the proof.

This estimate is one of the basic ingredients of the proof of the space-time Strichartz estimates*:*

$$\|u\|_{\mathrm{L}^p(\mathbb{R}_+,\mathrm{L}^q(\mathbb{R}^d))} \leq C(p,q,d)\, \|f\|_{\mathrm{L}^2(\mathbb{R}^d)}$$

for all $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$*,* $(p,q) \neq (2,\infty)$*,* $p \geq 2$*.*

Proof. (1) g_α is a bounded function (of norm equal to 1), so

$$|\langle g_\alpha, \varphi\rangle| \leq \|\varphi\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = \|\varphi\|_{0,0}\,,$$

which shows that g_α is a tempered distribution. By analytic continuation, the formula (**??**) is for all complex values $\alpha \in \mathbb{C}$ such that $e^{-\alpha|x|^2}$ is a tempered distribution (that is, for all $\alpha \in C \cap \{\alpha : \mathrm{Re}\,(\alpha) \geq 0\}$). Replacing α by $i\,\alpha$ shows that

$$\mathscr{F}(x \mapsto e^{-i\,\alpha|x|^2})(\xi) = \left(\frac{\pi}{i\,\alpha}\right)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4\,i\,\alpha}} = \left(\frac{\pi}{i\,\alpha}\right)^{\frac{d}{2}} e^{i\frac{|\xi|^2}{4\,\alpha}}.$$

(2) Taking the Fourier transform in $x \in \mathbb{R}^d$ of the equation, we find

$$i\frac{\partial}{\partial t}\widehat{u}(t,\xi) - |\xi|^2\widehat{u}(t,\xi) = 0,$$

or

$$\frac{\partial}{\partial t}\widehat{u}(t,\xi) = -i|\xi|^2\widehat{u}(t,\xi).$$

Therefore, we get

$$\widehat{u}(t,\xi) = e^{-i\,t\,|\xi|^2}\widehat{u}(0,\xi) = e^{-i\,t\,|\xi|^2}\,\widehat{f}(\xi). \tag{6}$$

Now, by the Fourier inverse formula and the first question, we have

$$\frac{1}{(2\pi)^d}\mathscr{F}\left(\frac{1}{(2\pi)^d} \times \left(\frac{\pi}{i\,\alpha}\right)^{\frac{d}{2}} e^{i\frac{|x|^2}{4t}}\right) = e^{-i\,t\,|\xi|^2},$$

which shows thanks to the convolution identity $\mathscr{F}(\varphi * \psi) = \mathscr{F}(\varphi)\mathscr{F}(\psi)$ valid for all $\varphi \in \mathscr{S}'(\mathbb{R}^d)$ and $\psi \in \mathscr{S}(\mathbb{R}^d)$ that

$$u(t,x) = \left(y \mapsto \frac{1}{(2\pi i t)^{\frac{d}{2}}} e^{i\frac{|y|^2}{4t}}\right) * f(x) = \frac{1}{(2\pi i t)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} f(y)dy.$$

(3) By Parseval identity and (6), we have

$$\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = \frac{1}{(2\pi)^{\frac{d}{2}}}\,\|\widehat{u}\|_{\mathrm{L}^2(\mathbb{R}^d)} = \frac{1}{(2\pi)^{\frac{d}{2}}}\left\|\widehat{f}\right\|_{\mathrm{L}^2(\mathbb{R}^d)} = \|f\|_{\mathrm{L}^2(\mathbb{R}^d)}\,,$$

while the triangle inequality directly implies that

$$|u(t,x)| \leq \frac{1}{(2\pi t)^{\frac{d}{2}}}\int_{\mathbb{R}^d} |f(y)|dy = \frac{1}{(2\pi t)^{\frac{d}{2}}}\,\|f\|_{\mathrm{L}^1(\mathbb{R}^d)}\,.$$

(4) We notice that u_λ is also a solution of (5) of initial data $f_\lambda(x) = f(\lambda x)$ since

$$\partial_t u_\lambda(t,x) = \lambda^2\,\partial_t u(\lambda^2 t, \lambda\, x)$$

$$\partial_{x_j}^2 u_\lambda(t,x) = \lambda^2 \partial_{x_j}^2 u(\lambda^2 t, \lambda x)$$
$$\Delta u_\lambda(t,x) = \lambda^2 \Delta u(\lambda^2 t, tx).$$

Now, we compute

$$\|u_\lambda(t,\,\cdot\,)\|_{\mathrm{L}^p(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |u(\lambda^2 t, \lambda x)|^p dx\right)^{\frac{1}{p}} \underset{\lambda x = y}{=\!=} \frac{1}{\lambda^{\frac{d}{p}}} \left(\int_{\mathbb{R}^d} |u(\lambda^2 t, y)| dy\right)^{\frac{1}{p}}.$$

Likewise, we have

$$\|f_\lambda\|_{\mathrm{L}^{p'}(\mathbb{R}^d)} = \frac{1}{\lambda^{\frac{d}{p'}}} \|f\|_{\mathrm{L}^2(\mathbb{R}^d)}.$$

Finally, we deduce that

$$\left\|u(\lambda^2 t,\,\cdot\,)\right\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq \frac{C}{t^\alpha} \frac{1}{\lambda^{d\left(\frac{1}{p'}-\frac{1}{p}\right)}} \|f\|_{\mathrm{L}^{p'}(\mathbb{R}^d)}.$$

Choosing $\lambda = t^{-2}$, we get

$$\|u(1,\cdot\,)\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq \frac{C}{t^{\alpha - \frac{d}{2}\left(\frac{1}{p'}-\frac{1}{p}\right)}} \|f\|_{\mathrm{L}^{p'}(\mathbb{R}^d)}.$$

Taking $t \to \infty$ or $t \to 0$, we deduce that

$$\alpha = \frac{d}{2}\left(\frac{1}{p'} - \frac{1}{p}\right) = \frac{d}{2}\left(1 - \frac{2}{p}\right) = d\left(\frac{1}{2} - \frac{1}{p}\right).$$

□

Exercise 3 (Elliptic Regularity)**.** This exercise aims at generalising the *elliptic estimates* on the harmonic functions. Namely, $\Delta u = f \in C^\infty(\Omega)$ implies that $u \in C^\infty(\Omega)$. Here, we will prove the main step that shows that weak solution of elliptic partial differential equation are continuous.

Let $\Omega \subset \mathbb{R}^d$ be a connected open subset. Recall that for all $1 \leq p, q \leq \infty$, for all $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ and $\varphi \in W_0^{1,q}(\Omega)$ such that $u\nabla\varphi \in L^1(\Omega)$ and $\varphi\nabla u \in L^1(\Omega)$, we have for all $1 \leq j \leq d$

$$\int_\Omega u\,\partial_{x_j}\varphi\,dx = -\int_\Omega \left(\partial_{x_j} u\right)\varphi\,dx.$$

Restricting to $\Omega = B(0,1)$ from now on, we let $A = (a_{i,j})_{1\leq j\leq d} \in L^\infty(B(0,1), \mathbb{R}^d)$ be a space-dependent *uniformly elliptic* matrix, *i.e.* there exists $0 < \Lambda < \infty$ such that

$$\Lambda^{-1}|\xi|^2 \leq \xi^t A(x)\xi = \sum_{i,j=1}^d a_{i,j}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \forall x \in B(0,1), \forall \xi \in \mathbb{R}^d. \tag{7}$$

We now let $u \in W^{1,2}(B(0,1))$ a weak solution (in $\mathscr{D}'(B(0,1))$) of the linear partial differential equation

$$\operatorname{div}\left(A(x)\nabla u\right) = 0. \tag{8}$$

Important note: all inequalities can be proven with "worse" constants than stated, provided that they are universal constants (only depend on the ambient dimension and the parameter Λ), unless stated otherwise.

(1) For $d = 1$, show directly that $u \in C^0(B(0,1))$.

Proof. It is a direct consequence of the Sobolev embedding $W^{1,2}(]0,1[) \hookrightarrow C^{0,\frac{1}{2}}([0,1])$. □

(2) Show that for all $\varphi \in \mathscr{D}(B(0,1)) = C_c^\infty(B(0,1))$, the following identity holds

$$\int_{B(0,1)} A(x)\nabla u \cdot \nabla\varphi \, dx = 0.$$

Notice that for all $u, v \in H^1(B(0,1))$, we have

$$\int_{B(0,1)} A(x)\nabla u \cdot \nabla v \, dx = \sum_{i,j=1}^{d} \int_{B(0,1)} a_{i,j}(x)\partial_{x_i} u \, \partial_{x_j} v \, dx.$$

Proof. Since $\operatorname{div}(A(x)\nabla u) = 0$, by multiplying this equation by φ and integrating by parts, the identity follows immediately. □

(3) (Caccioppoli inequality) Let $u \in W^{1,2}(B(0,1))$. Show that for all $\eta \in \mathscr{D}(B(0,1))$, and $c \in \mathbb{R}$, we have

$$\int_{B(0,1)} |\nabla u|^2\eta^2 dx \leq \Lambda^2 \int_{B(0,1)} |u-c|^2|\nabla\eta|^2 dx.$$

One can use the following test function $\varphi = (u-c)\eta^2$.

Proof. We have

$$0 = \int_{B(0,1)} A(x)\nabla u \cdot \nabla\varphi \, \eta^2 \, dx = \int_{B(0,1)} A(x)\nabla u \cdot \nabla u \, \eta^2 \, dx + 2\int_{B(0,1)} A(x)\nabla u \cdot (\nabla\eta)\,(u-c)\eta \, dx.$$

Using the elliptic estimate, we have

$$\Lambda^{-1}\int_{B(0,1)} |\nabla u|^2 dx \leq \int_{B(0,1)} A(x)\nabla u \cdot \nabla u \, \eta^2 \, dx.$$

Now, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left|\int_{B(0,1)} A(x)\nabla u \cdot (\nabla\eta)(u-c)\eta \, dx\right| &\leq \Lambda \int_{B(0,1)} |\nabla u||\nabla\eta||\eta| dx \\ &\leq \Lambda\left(\int_{B(0,1)} |\nabla u|^2\eta^2 dx\right)^{\frac{1}{2}}\left(\int_{B(0,1)} |u-c|^2 dx\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore, we have

$$\Lambda^{-1}\int_{B(0,1)} |\nabla u|^2\eta^2 \, dx \leq \Lambda\left(\int_{B(0,1)} |\nabla u|^2\eta^2 dx\right)^{\frac{1}{2}}\left(\int_{B(0,1)} |u-c|^2|\nabla\eta|^2 dx\right)^{\frac{1}{2}}.$$

Since all integrals are finite, if the left-hand side integral is zero, then we are done. Otherwise, we divide by the square root of the left-hand side and find that

$$\int_{B(0,1)} |\nabla u|^2\eta^2 \, dx \leq \Lambda^2 \int_{B(0,1)} |u-c|^2|\nabla\eta|^2 dx$$

□

(4) By choosing an appropriate test function η, show that for all $0 < r_1 < r_2 < 1$, we have for some universal constant $0 < C < \infty$,

$$\int_{B(0,r_1)} |\nabla u|^2 dx \leq \frac{C}{(r_2-r_1)^2}\int_{B(0,r_2)\setminus\overline{B}(0,r_1)} |u-c|^2 dx \tag{9}$$

This is the *Caccioppoli inequality.*

Proof. We take η such that $\eta = 1$ on $B_{r_1}(0)$, and $\text{supp}(\eta) \subset B(0, r_2)$. By using an approximation of a linear function, we can choose η radial such that $|\nabla \eta| \leq \frac{2}{(r_2 - r_1)}$, which shows by the previous inequality since $\nabla \eta = 0$ on $B(0, r_1)$ that

$$\int_{B_{r_1}(0)} |\nabla u|^2 dx \leq \int_{B_{r_1}(0)} |\nabla u|^2 \eta^2 dx \leq \frac{4\Lambda^2}{(r_2 - r_1)^2} \int_{B_{r_2}(0)} |u - c|^2 dx.$$

□

(5) Using the Poincaré-Wirtinger inequality and a scaling argument, show that for all for all $0 < r < \infty$ if $A(r) = B(0, 2r) \setminus \overline{B}(0, r)$ and $u \in W^{1,2}(B(0, r))$, we have

$$\int_{A(r)} |u - u_r|^2 dx \leq C r^2 \int_{A(r)} |\nabla u|^2 dx,$$

where $u_r = ⨍_{A(r)} u\, dx = \dfrac{1}{c_d\, r^d} \displaystyle\int_{A(r)} u\, dx$ (for some universal constant c_d) is the mean of u on $A(r)$, and $0 < C < \infty$ is a universal constant independent of r.

Proof. By the Poincaré-Wirtinger inequality, we have

$$\int_{A(1)} |v - v_1|^2 dx \leq C \int_{A(r)} |\nabla v|^2 dx$$

for all $v \in W^{1,2}(A(1))$. If $v(x) = u(rx)$, where $u \in W^{1,2}(A(r))$, then by the change of variable $rx = y$, we get

$$\int_{A(1)} |v - v_1|^2 dx = r^{-d} \int_{A(r)} |u - u_r|^2 dy,$$

while

$$\int_{A(1)} |\nabla v|^2 dx = \int_{A(1)} r^2 |\nabla u(rx)|^2 dx = r^{2-d} \int_{A(r)} |\nabla u|^2 dy,$$

which concludes the proof by multiplying both sides by r^d. □

(6) Deduce from the previous questions that there exists a universal contant $0 < \theta < 1$ such that for all $0 < r < \dfrac{1}{2}$, we have

$$\int_{B(0,r)} |\nabla u|^2 dx \leq \theta \int_{B(0,2r)} |\nabla u|^2 dx.$$

By induction, deduce that there exists $0 < \alpha < 1$ and $0 < C < \infty$ such that for all $0 < r < \dfrac{1}{2}$, we have

$$\int_{B(0,r)} |\nabla u|^2 \leq C r^{\alpha} \int_{B(0,1)} |\nabla u|^2 dx.$$

Hint: use a dyadic argument.

Proof. Using the Caccioppoli inequality and the previous Poincaré inequality, we get

$$\int_{B(0,r)} |\nabla u|^2 dx \leq \frac{\Lambda^2}{r^2} \times C r^2 \int_{B(0,2r) \setminus \overline{B}(0,r)} |\nabla u|^2 dx = C\Lambda^2 \int_{B(0,2r) \setminus \overline{B}(0,r)} |\nabla u|^2 dx.$$

By adding $C\Lambda^2 \int_{B(0,r)} |\nabla u|^2 dx$ on both sides, we get

$$(1+C\Lambda^2)\int_{B(0,r)} |\nabla u|^2 dx \leq C\Lambda^2 \int_{B(0,2r)} |\nabla u|^2 dx,$$

which proves the inequality with $\theta = \frac{C\Lambda^2}{1+C\Lambda^2} \in (0,1)$.

Let $j \in \mathbb{N}$ such that $2^j r < 1 \leq 2^{j+1} r$. The inequality shows that $j\log(2) + \log(r) < 0 \leq (j+1)\log(2) + \log(r)$, or

$$j < \log_2\left(\frac{1}{r}\right) \leq j+1,$$

and

$$\log_2(r) < -j \leq 1 + \log_2(r).$$

Then, by a direct induction, we deduce that

$$\int_{B(0,r)} |\nabla u|^2 dx \leq \theta^j \int_{B(0,2^j r)} |\nabla u|^2 dx \leq \frac{1}{\theta} r^{\log_2\left(\frac{1}{\theta}\right)} \int_{B(0,1)} |\nabla u|^2 dx,$$

since $\theta^j = \exp\left(-j\log\left(\frac{1}{\theta}\right)\right) \leq \exp\left((1+\log_2(r))\log\left(\frac{1}{\theta}\right)\right) = \frac{1}{\theta} r^{\log_2\left(\frac{1}{\theta}\right)}$. We deduce the claim with $\alpha = \log_2\left(\frac{1}{\theta}\right)$. □

(7) With the help of another optimal Poincaré-Wirtinger inequality, deduce that there exists a universal constant $0 < C < \infty$ such that for all $0 < r < \dfrac{1}{2}$,

$$\int_{B(0,r)} |u - \widetilde{u}_r|^2 dx \leq C r^{2+\alpha} \int_{B(0,1)} |\nabla u|^2 dx,$$

where $\widetilde{u}_r = \fint_{B(0,r)} u\, dx = \dfrac{1}{c_d' r^d} \int_{B(0,r)} u\, dx$ is the mean of u on $B(0,r)$.

Proof. The proof is exactly the same as before and we omit it. □

(8*) In the special case $d = 2$, show with the help of Lebesgue differentiation theorem and a translation argument that $u \in C^{0,\beta}(B(0,\frac{1}{4}))$ for some $0 < \beta < 1$. Recall that the Lebesgue differentiation theorem shows that for all $u \in L^1_{\text{loc}}(B(0,1))$, and for all $x \in B(0,1)$, we have[‡]

$$\fint_{B(x,r)} u(y)\, dy = \frac{1}{\pi r^2} \int_{B(x,r)} u(y)\, dy \underset{r\to 0}{\longrightarrow} u(x).$$

Furthermore, recall that the space of β-Hölder functions is defined by

$$C^{0,\beta}(\Omega) = C^0(\Omega) \cap \left\{ u : \|u\|_{C^{0,\beta}(\Omega)} = \|u\|_{\mathrm{L}^\infty(\Omega)} + \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\beta} < \infty \right\}$$

and equipped with the $\|u\|_{C^{0,\beta}(\Omega)}$ norm defined above. On may first prove the following estimate

$$|u_{x_0,r} - u_{x_0,s}| \leq C \frac{r^{1+\frac{\alpha}{2}}}{s} \int_{B(0,1)} |\nabla u|^2 dx$$

for all $x_0 \in B(0,\frac{1}{2})$, and $0 < s < r < \frac{1}{2}$, where $u_{x_0,r} = \fint_{B(x_0,r)} u\, dx = \dfrac{1}{\pi r^2} \int_{B(x_0,r)} u\, dx$.

[‡]Notice that the theorem is stated on $\mathbb{R}^2$.

Proof. By translation, we deduce that for all $x_0 \in B(0, \frac{1}{4})$, we have for all $0 < r < \frac{1}{2}$

$$\int_{B(x_0,r)} |u - u_{x_0,r}|^2 dx \leq C r^{2+\alpha} \left(\int_{B(0,1)} |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$

Now, we have by the triangle inequality

$$\begin{aligned}
|u_{x_0,r} - u_{x_0,s}|^2 &= \fint_{B(x_0,s)} |u_{x_0,r} - u_{x_0,s}|^2 dx \\
&\leq 2 \fint_{B(x_0,s)} |u - u_{x_0,r}|^2 dx + 2 \int_{B(x_0,s)} |u - u_{x_0,s}|^2 dx \\
&\leq C \frac{r^2}{s^2} \fint_{B(x_0,r)} + C s^\alpha \int_{B(0,1)} |\nabla u|^2 dx \\
&\leq C \left(\frac{r^{2+\alpha}}{s^2} + \frac{s^{2+\alpha}}{s^2} \right) \int_{B(0,1)} |\nabla u|^2 dx \\
&\leq C \frac{r^{2+\alpha}}{s^2} \int_{B(0,1)} |\nabla u|^2 dx,
\end{aligned}$$

which proves the inequality. Now, take $r = r_i = r_0 2^{-i}$, and $s = r_{i+1} = r_0 2^{-i-1}$. We get

$$|u_{x_0,r_i} - u_{x_0,r_{i+1}}| \leq C 2^{-\frac{\alpha}{2} i} r_0^\alpha \|\nabla u\|_{L^2(B(0,1))}.$$

Therefore, the series $\sum_{i=0}^{\infty} |u_{x_0,r_i} - u_{x_0,r_{i+1}}|$ is summable, and we deduce in particular that $u_{x_0,r_i} \underset{i\to\infty}{\longrightarrow} u(x_0)$ by Lebesgue differentiation theorem. Furthermore, we have

$$|u(x_0) - u_{x_0,r_0}| \leq C r_0^{\frac{\alpha}{2}} \|\nabla u\|_{L^2(B(0,1))}.$$

Finally, if $x_0, y_0 \in B(0, \frac{1}{4})$ and $r_0 = 2|x_0 - y_0|$, we have by the triangle inequality

$$|u(x_0) - u(y_0)| \leq C|x_0 - y_0|^{\frac{\alpha}{2}} \|\nabla u\|_{L^2(B(0,1))} + |u_{x_0,r_0} - u_{y_0,r_0}|,$$

and

$$\begin{aligned}
|u_{x_0,r_0} - u_{y_0,r_0}|^2 &\leq \fint_{B(y_0,\frac{r_0}{2})} |u_{x_0,r_0} - u_{y_0,r_0}|^2 dx \\
&\leq 8 \fint_{B(x_0,r_0)} |u - u_{x_0,r_0}|^2 dx + 8 \int_{B(y_0,r_0)} |u - u_{y_0,r_0}|^2 dx \\
&\leq C|x_0 - y_0|^\alpha,
\end{aligned}$$

which concludes the proof. □

(9) Assume from now on that $d \geq 3$. We say that $u \in H^1(B(0,1))$ is a weak sub-solution if for all $\varphi \in \mathscr{D}(B(0,1))$, we have

$$\int_\Omega A(x) \nabla u \cdot \nabla \varphi \, dx \leq 0, \tag{10}$$

or in other words

$$\sum_{i,j=1}^{d} \int_\Omega a_{i,j}(x) \partial_{x_i} u \, \partial_{x_j} v \, dx \leq 0.$$

Using the chain rule and the product formula for Sobolev functions, show that for all convex, increasing function $\Phi \in C^2(\mathbb{R}, \mathbb{R}_+)$ such that $\Phi''(t) = 0$ for all $|t| \geq R$ (for some fixed $R > 0$), then for any sub-solution $u \in H^1(B(0,1))$, the function $\Phi(u) \in H^1(B(0,1))$ is a sub-solution.

From now on, we assume that the previous result holds for any increasing, convex function $\Phi : \mathbb{R} \to \mathbb{R}_+$ (not necessarily C^2 or satisfying the extra hypothesis on the second derivative).

(10) (Moser iterations) Fix some $0 < 2r < 1$, and for all $j \in \mathbb{N}$, let $B_j = B(0, r + 2^{-j}r)$, so that $B_{j+1} \subset B_j \subset B(0, 2r)$ and

$$B_\infty = \bigcap_{j \in \mathbb{N}} B_j = B(0, r).$$

Show that for all $1 \leq \lambda \leq \dfrac{d}{d-2}$, there exists $0 < \gamma < \infty$ such that for all $j \in \mathbb{N}$, for all $v \in H^1(B_{j+1})$, we have

$$\left\||v|^\lambda\right\|_{\mathrm{L}^2(B_{j+1})}^2 \leq \gamma \left\|\nabla v\right\|_{\mathrm{L}^2(B_{j+1})}^{2\lambda} + \gamma \left\|v\right\|_{\mathrm{L}^2(B_{j+1})}^{2\lambda}.$$

Proof. By the Sobolev embedding theorem, since $2^* = \frac{2d}{d-2}$, there exists $0 < C = C(r) < \infty$ (independent of j by a scaling argument since $B(0,r) \subset B_j \subset B(0, 2r)$ for all $j \in \mathbb{N}$) and such that

$$\begin{aligned} \left\||v|^\lambda\right\|_{\mathrm{L}^2(B_{j+1})}^2 &= \left\|v\right\|_{\mathrm{L}^{2\lambda}(B_{j+1})}^{2\lambda} \leq C \left\|v\right\|_{\mathrm{W}^{1,2}(B_{j+1})}^{2\lambda} = C \left(\left\|\nabla v\right\|_{\mathrm{L}^2(B_{j+1})} + \left\|v\right\|_{\mathrm{L}^2(B_j)}\right)^{2\lambda} \\ &\leq 2^{2\lambda - 1} C \left(\left\|\nabla v\right\|_{\mathrm{L}^2(B_{j+1})}^{2\lambda} + \left\|v\right\|_{\mathrm{L}^2(B_{j+1})}^{2\lambda}\right) \end{aligned}$$

by convexity of $\mathbb{R}_+ \to \mathbb{R}_+, t \mapsto t^{2\lambda}$. ☐

(11) Fix $1 < \lambda \leq \frac{d}{d-2}$, and let $u \in H^1(B_j)$ a *positive* weak sub-solution. With the help of the Caccioppoli inequality (explain why it holds true for a sub-solution), show that $u^\lambda \in H^1(B_{j+1})$, that u^λ is a positive weak sub-solution, and show the inequality

$$\left\|u^\lambda\right\|_{\mathrm{L}^2(B_{j+1})}^2 \leq C(2^{2\lambda j} + 1) \left\|u\right\|_{\mathrm{L}^2(B_j)}^{2\lambda},$$

where C only depends on $d, \Lambda, r > 0$.

Proof. By (10), $u^\lambda \in L^2(B_{j+1})$, and by the extension of question (9) to general convex functions, we deduce that u^λ is a positive weak sub-solution. By Caccioppoli inequality that works for sub-solutions since the proof works *mutadis mutandis* if one replace the equality by an inequality, we get

$$\left\|\nabla u\right\|_{\mathrm{L}^2(B_{j+1})} \leq \frac{2\Lambda}{(r + 2^{-j}r) - (r + 2^{-j-1}r)} \left\|u\right\|_{\mathrm{L}^2(B_j)} = \frac{4\Lambda}{r} 2^j \left\|u\right\|_{\mathrm{L}^2(B_j)}..$$

Since $\left\|u\right\|_{\mathrm{L}^2(B_{j+1})} \leq \left\|u\right\|_{\mathrm{L}^2(B_j)}$, we get by (10)

$$\begin{aligned} \left\|u^\lambda\right\|_{\mathrm{L}^2(B_{j+1})}^2 &\leq \gamma \left(1 + \frac{4\Lambda}{r} 2^{2\lambda j}\right) \left\|u\right\|_{\mathrm{L}^2(B_j)}^{2\lambda} \\ &= C(r)(2^{2\lambda j} + 1) \left\|u\right\|_{\mathrm{L}^2(B_j)}^{2\lambda}. \end{aligned}$$

☐

(12) Let $u \in H^1(B(0,1))$ be a positive weak sub-solution. Define for all $j \in \mathbb{N}$ the function $u_j = u^{\lambda^j}$. Explain why

$$\left\|u\right\|_{\mathrm{L}^\infty(B(0,r))} = \limsup_{j \to \infty} \left\|u\right\|_{\mathrm{L}^{2\lambda^j}(B_j)},$$

and prove the following inequality

$$\left\|u\right\|_{\mathrm{L}^\infty(B(0,r))} \leq C(d, \Lambda, r) \left\|u\right\|_{\mathrm{L}^2(B(0,2r))}.$$

Proof. This is a standard result from the theory of L^p spaces. Now, notice that the previous inequality can be rewritten as

$$\|u_{j+1}\|_{\mathrm{L}^2(B_{j+1})} \leq C(2^{2\lambda j}+1)\,\|u_j\|^{\lambda}_{\mathrm{L}^2(B_j)}\,.$$

Since $L_{j+1} = \|u\|_{\mathrm{L}^{2\lambda^{j+1}}(B_{j+1})} = \|u_{j+1}\|^{\frac{1}{\lambda^{j+1}}}_{\mathrm{L}^2(B_{j+1})}$ Now, we have by direct induction

$$L_{j+1} \leq \left(C(2^{2\lambda j}+1)\right)^{\frac{1}{\lambda^{j+1}}} L_j,$$

which shows by a direct induction that

$$L_j \leq \prod_{i=0}^{j} \left(C(2^{2\lambda i}+1)\right)^{\frac{1}{\lambda^{i+1}}} L_0 = \prod_{i=0}^{j} \left(C(2^{2\lambda i}+1)\right)^{\frac{1}{\lambda^{i+1}}} \|u\|_{\mathrm{L}^2(B(0,1))}\,.$$

Since the product

$$\prod_{i=0}^{\infty} \left(C(2^{2\lambda i}+1)\right)^{\frac{1}{\lambda^{i+1}}}$$

is absolutely convergent, we obtain the announced estimate. Indeed, notice that

$$\left(C(2^{2\lambda i}+1)\right)^{\frac{1}{\lambda^{i+1}}} = \exp\left(\frac{1}{\lambda^{i+1}}\log(1+2^{2\lambda i})\right) = \exp\left(2\lambda\log(2)\frac{i}{\lambda^{i+1}} + o\left(\frac{i}{\lambda^{i+1}}\right)\right)$$

and the geometric series

$$\sum_{i=0}^{\infty} \frac{i}{\lambda^{i+1}}$$

converges since $\lambda > 1$. □