Calculus of Variations

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Introduction

Calculus of variations consists in solving partial differential equations by minimising or maximising an energy, or more generally, by constructing critical points of a given functional. It goes back to the work of Euler from 1744 ([9]) for 1-dimensional problems and to the work of Lagrange from 1760 ([17]) for problems in arbitrary dimension. Despite the efforts of the most brilliant minds of the time, what became the Dirichlet principle (*Dirichletsche Prinzip*), namely, that one can construct a minimiser of any reasonable functional (in particular, of the Dirichlet energy), could not be justified rigorously until the work of Hilbert in 1904 ([13]), exactly 160 years after Euler's seminal work. Since Hilbert's fundamental contribution, the calculus of variations has developed in manifold ways, with a recent emphasis on min-max methods in cases where the standard methods fail (lack of coercivity of functionals, lack of compactness, etc). In these lecture notes, we will show how modern methods allow us to recover fairly easily Hilbert's results and several generalisations, and we will study in details a degenerate case (the problem of Plateau) that has had a tremendous influence on the entire field. We will start the lectures by explaining the need of finding the appropriate functional spaces (Sobolev spaces), and then delve into various notions of convexity where the so-called "direct method of the calculus of variations" works.

Chapter 1

General introduction

1.1 Early History and Sobolev Spaces

The modern study of polynomial equations begins with the proof of the d'Alembert-Gauss theorem in 1815. Their study had started during Antiquity (it is more ancient than the Babylonian civilisation). Contrary to his predecessors who were looking for explicit solutions expressible by a succession of square roots (which was bound to fail for equations of degree $n > 4$), Gauss showed by an abstract method that any polynomial with complex coefficients admits exactly n roots (with multiplicity)—and he proved this result more than a decade before the revolutionary work of Galois and Abel.

The calculus of variations was founded as a field by the successive contributions of Leonhard Euler in 1744 ([9]) and Joseph-Louis Lagrange in 1760 ([17])—for higher dimension problems (Euler had restricted his theory to 1-dimensional problems). The notion of Euler-Lagrange equation follows directly from their works and can be stated as follows: let $\Omega \subset \mathbb{R}^d$ be an open domain, $E : C^\infty(\Omega) \to \mathbb{R}$. Assume that $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ is such that $E(u) \leq E(v)$ for all $v \in C^\infty(\Omega)$ such that $v = u$ on $\partial\Omega$. Then, if E admits a directional derivative in the direction $h \in C^\infty(\Omega)$ such that $h = 0$ on $\partial\Omega$, then we have

$$D_h E(u) = \frac{d}{dt} E(u + t\,h) = 0. \tag{1.1.1}$$

Indeed, by definition of Taylor expansion, we have

$$E(u + t\,h) = E(u) + t\,D_h E(u) + o(t).$$

If $D_h E(u) \neq 0$ for $|t| > 0$ small enough (depending on the sign of $D_h E(u) \in \mathbb{R}^*$), we get a contradiction as $v_t = u + t\,h \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ is such that $v_t = u$ on $\partial\Omega$ for all $t \in \mathbb{R}$. Functions E whose domain is a function space are called functionals or Lagrangians. One of the simplest Lagrangians is given by the Dirichlet energy

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx = \frac{1}{2}\int_\Omega \sum_{i=1}^d \left(\frac{\partial u}{\partial x_i}\right)^2 dx.$$

We have

$$E(u + t\,h) = \frac{1}{2}\int_\Omega |\nabla u + t\,\nabla h|^2 dx = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + t\int_\Omega \nabla u \cdot \nabla h\,dx + \frac{t^2}{2}\int_\Omega |\nabla h|^2 dx.$$

Therefore, we have by Stokes formula

$$\begin{aligned} D_h E(u) &= \int_\Omega \nabla u \cdot \nabla h\,dx = \int_\Omega \mathrm{div}(h\nabla u)dx - \int_\Omega h\,\Delta u\,dx = \int_{\partial\Omega} h\,\partial_\nu u\,d\sigma - \int_\Omega h\,\Delta u\,dx \\ &= -\int_\Omega h\,\Delta u\,dx, \end{aligned}$$

where we used the hypothesis $h = 0$ on $\partial\Omega$. As the equation is satisfied for all function $h \in C^0(\overline{\Omega})$ such that $h = 0$ on $\partial\Omega$, we deduce that u satisfies the equation $\Delta u = 0$ in Ω, where Δ is the Laplacian operator, given by

$$\Delta = \mathrm{div}\,\nabla = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}.$$

We say that a function satisfying the equation $\Delta u = 0$ is a *harmonic* function. The Dirichlet problem consists in finding a harmonic function of prescribed boundary. Explicitly, if $f \in C^0(\partial\Omega)$, does there exist a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $\Delta u = 0$ in Ω and $u = f$ on $\partial\Omega$?

As in the case of polynomial equations, there exists explicit formulae for certain domains whose geometry is simple enough. The most famous formula holds for the unit disk $\mathbb{D} \subset \mathbb{C} \simeq \mathbb{R}^2$ defined by

$$\mathbb{D} = \mathbb{C} \cap \{z : |z| < 1\}.$$

If $f \in C^0(\partial\mathbb{D})$, then the function

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} f\left(e^{i\theta}\right) d\theta \tag{1.1.2}$$

is harmonic and is the only continuous solution to the equation $\Delta u = 0$ in $\mathbb{D}$ and $u = f$ on $\partial\mathbb{D} = S^1$. In an analogous way, there exists similar formulae for the d-dimensional unit ball. Explicitly, if $x_0 \in \mathbb{R}^d$, $r > 0$, and if we define the radius r ball of centre x_0 by

$$B(x_0, r) = \mathbb{R}^d \cap \{x : |x - x_0| < r\},$$

then for every harmonic function u in $B(x_0, r)$, we have

$$u(x) = \frac{1}{\beta(d)} \int_{\partial B(x_0,r)} \frac{r^2 - |x-x_0|^2}{r|x-y|^2} u(y) d\mathscr{H}^{d-1}(y), \tag{1.1.3}$$

where $\mathscr{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure (or alternatively, the standard volume form on the sphere $\partial B(x_0, r)$) and $\beta(d) = \mathscr{H}^{d-1}(S^{d-1})$ is the measure of the unit sphere $S^{d-1} = \partial B(0,1)$*. Poisson's formula (1.1.2) was already well known of Gauss and Kelvin, and allowed one to solve the Poisson's problem in sufficiently simple domains. In particular, in the case of 2-dimensional domains, the uniformisation theorem of Riemann allows one to solve the Dirichlet problem in any simply connected domain. Let us recall the statement of this theorem.

Theorem 1.1.1 (Riemann)**.** *Let $\Omega \subset \mathbb{C}$ be a simply connected domain. Then, there exists a biholomorphic map $\varphi : \Omega \to \mathbb{D}$ from Ω into the unit disk $\mathbb{D}$.*

Furthermore, the Cauchy-Riemann equations show that φ is a conformal map. If $\varphi = f + i\,g$, then we can identify φ with the map $\psi = (f, g) : \Omega \to \mathbb{R}^2$. As φ is holomorphic, we have

$$0 = \partial_{\overline{z}}\varphi = \frac{1}{2}\left(\partial_x + i\,\partial_y\right)(f + i\,g) = \frac{1}{2}\left(\partial_x f - \partial_y g + i\left(\partial_y f + \partial_x g\right)\right).$$

Therefore, we deduce that

$$|\partial_x\psi|^2 = (\partial_x f)^2 + (\partial_x g)^2 = (\partial_y g)^2 + (-\partial_y f)^2 = |\partial_y\psi|^2 = |\nabla f|^2 = |\nabla g|^2,$$

and

$$\langle \partial_x\psi, \partial_y\psi\rangle = \partial_x f\,\partial_y f + \partial_x g\,\partial_y g = \partial_x f\,\partial_y x - \partial_y f\,\partial_x f = 0,$$

*Explicitly, we have $\beta(d) = \dfrac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$, where Γ is Euler's Gamma function ([11, **3.2.13**]).

which shows indeed that ψ is a conformal map. Recall that if $\Omega \subset \mathbb{R}^2$ is an open domain, we say that a map $f \in C^1(\Omega, \mathbb{R}^2)$ is conformal if

$$|\partial_x f| = |\partial_y f| > 0 \qquad \text{and} \qquad \langle \partial_x f, \partial_y f \rangle = 0.$$

This condition implies that f infinitesimally preserves the angles.

In particular, the Jacobian determinant of ψ is given by

$$\mathrm{Jac}(\psi) = \det \nabla\psi = \det \begin{pmatrix} \partial_x f & \partial_x g \\ \partial_y f & \partial_y g \end{pmatrix} = \det \begin{pmatrix} \partial_x f & -\partial_y f \\ \partial_y f & \partial_x f \end{pmatrix} = (\partial_x f)^2 + (\partial_y f)^2 = |\nabla f|^2. \tag{1.1.4}$$

If $v = u \circ \psi$, the chain rule implies that

$$\begin{aligned} \partial_x v &= \partial_x f\, \partial_x u + \partial_x g\, \partial_y u = \partial_x f\, \partial_x u - \partial_y f\, \partial_y u \\ \partial_y v &= \partial_y f\, \partial_x u + \partial_y g\, \partial_y u = \partial_y f\, \partial_x u + \partial_x f\, \partial_y u. \end{aligned}$$

Therefore, we deduce that

$$|\nabla v|^2 = |\partial_x f\, \partial_x u - \partial_y f\, \partial_y u|^2 + |\partial_y f\, \partial_x u + \partial_x f\, \partial_y u|^2 = |\nabla f|^2 |\nabla u|^2,$$

and the change of variable formula shows that

$$\int_{\psi^{-1}(\Omega)} |\nabla v|^2 dx = \int_{\psi^{-1}(\Omega)} |\nabla u \circ \psi|^2 |\mathrm{Jac}(\psi)| dx = \int_{\Omega} |\nabla u|^2 dy.$$

As a consequence, the Dirichlet energy is conformally invariant, which shows by Poisson's formula that one can solve the Dirichlet energy on any simply connected domain of the plane. However, this method does not work for more complicated domain (and in higher dimension), so we have to renounce having explicit formulae. We can either use an approach with Green's functions (which do not always exist for more involved problems), or use a variational method. This latter approach will be the main focus of this course. Following a counter-example of Weierstrass from 1869, Hilbert proved in 1900 the existence of a solution to Dirichlet's problem by minimising the Dirichlet energy ([13]). This method, known under the name of *the direct method of the calculus of variations*, has been generalised in several settings ([6, 7, 4, 5]), notably by Courant and his school ([3]). In this course, it will be discussed at lengths.

1.2 The Problem of Plateau

We will see in the course that thanks to modern tools, it is relatively easy to solve the Dirichlet problem, and a main feature of the calculus of variations is to solve problems for which the favourable structure of the Dirichlet energy is not present. There are no universal methods and instead of trying to sketch a general theory, we will present the resolution of a major problem that had a tremendous influence on the calculus of variations: the Plateau's problem.

The problem of Plateau is however intimately linked to the Dirichlet problem. Proposed by the Belgian physicist Joseph Plateau (1849 and 1873), it consists in constructing a minimal surface, or surface that minimises the area, of prescribed boundary. Jesse Douglas was awarded one of the first two Fields medals in 1936 for his general solution of Plateau's problem ([8]; Tibor Radó solved the problem first, but his solution was less general; [18]). Let $\Gamma \subset \mathbb{R}^3$ be a closed simple curve and $\gamma : S^1 \to \mathbb{R}^3$ be a parametrisation of Γ. Then, a solution to the problem of Plateau is an immersion $\Phi : \mathbb{D} \to \mathbb{R}^3$ such that $\Phi = \gamma$ on $\partial\mathbb{D}$ and that minimises the area. Let us recall that an immersion is a C^1 map such that

$$|\partial_x \Phi \times \partial_y \Phi| > 0.$$

In other words, an immersion is a function that maps any pair of non-collinear vectors into (another) pair of non-collinear vectors. Experimentally, using soapy water and metal wires, one can construct a solution to Plateau's problem , but it does not imply of course that the underlying mathematical problem is solvable. The equation of minimal surfaces—for graphical solutions $\Phi(x, y) = (x, y, u(x, y))$—is the first example Lagrange gave of his method in 1760. The area functional is given in general (this expression comes from the change of variable formula for domains of different dimensions, also known under the name of *area formula* [11, **3.2.3**]) by

$$\mathrm{Area}(\Phi)=\int_{\mathbb{D}}\sqrt{|\partial_x\Phi|^2|\partial_y\Phi|^2-\langle\partial_x\Phi,\partial_y\Phi\rangle^2}dx\,dy.$$

For a graphical function as above, the area functional becomes

$$\mathrm{Area}(\Phi)=\int_{\mathbb{D}}\sqrt{1+|\nabla u|^2}dx\,dy.$$

For a graphical variation $\vec{\Psi}(x,y)=(x,y,h(x,y))$ where $h=0$ on $\partial\mathbb{D}$, we get

$$\begin{aligned}\mathrm{Area}(\Phi+t\,\vec{\Psi})&=\int_{\mathbb{D}}\sqrt{1+|\nabla(u+h)|^2}dx\,dy=\int_{\mathbb{D}}\sqrt{1+|\nabla u|^2+2t\langle\nabla u,\nabla h\rangle+t^2|\nabla h|^2}dx\,dy\\&=\int_{\mathbb{D}}\sqrt{1+|\nabla u|^2}\sqrt{1+\frac{2\langle\nabla u,\nabla h\rangle}{1+|\nabla u|^2}+\frac{|\nabla h|^2}{1+|\nabla u|^2}}dx\,dy\\&=\int_{\mathbb{D}}\sqrt{1+|\nabla u|^2}dx\,dy+t\int_{\mathbb{D}}\frac{\langle\nabla u,\nabla h\rangle}{\sqrt{1+|\nabla u|^2}}dx\,dy+O\left(t^2\right),\end{aligned}$$

where we used the formula

$$\sqrt{1+x}=1+\frac{1}{2}x+O(x^2).$$

We deduce that

$$\begin{aligned}D_{\vec{\Psi}}\mathrm{Area}(\Phi)&=\int_{\mathbb{D}}\frac{\langle\nabla u,\nabla h\rangle}{\sqrt{1+|\nabla u|^2}}dx\,dy=\int_{\partial\mathbb{D}}\frac{h\,\partial_\nu u}{\sqrt{1+|\nabla u|^2}}d\theta-\int_{\Omega}h\ \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)dx\,dy\\&=-\int_{\Omega}h\ \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)dx\,dy\end{aligned}$$

As the equation is satisfied for every function h that vanishes on the boundary, Stokes theorem shows that

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0\qquad\text{sur }\ \Omega.\tag{1.2.1}$$

In local coordinates (x,y), the equation can be rewritten as follows:

$$\left(1+\left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial^2u}{\partial x^2}-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2u}{\partial x\partial y}+\left(1+\left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2u}{\partial y^2}=0.\tag{1.2.2}$$

This equation is elliptic and *non-linear* (the coefficients of the equation are variables too). In order to solve it, direct methods are inefficient, but a rewriting of the equation will allow us to solve Plateau's problem. First, notice that by Cauchy's inequality $2\,ab\leq a^2+b^2$ $(a,b\in\mathbb{R})$, we have

$$\mathrm{Area}(\Phi)\leq\int_{\mathbb{D}}|\partial_x\Phi||\partial_y\Phi|dx\,dy\leq\frac{1}{2}\int_{\mathbb{D}}\left(|\partial_x\Phi|^2+|\partial_y\Phi|^2\right)dx\,dy=\frac{1}{2}\int_{\mathbb{D}}|\nabla\Phi|^2dx\,dy=E(\Phi).\tag{1.2.3}$$

Furthermore, both functional coincide if and only if Φ is conformal. The classical approach to solve the problem of Plateau is to find a conformal and harmonic (for a conformal map is minimal if and only if it is harmonic) map that satisfies suitable hypotheses on the boundary. The main difficulty of this approach is the lack of compactness. This is the first issue one must needs solve in the calculus of variations: it is necessary to find a class of functions *stable* (in a sense) under weak convergence. Indeed, if we choose a minimising sequence $\{\Phi_k\}_{k\in\mathbb{N}}\subset C^\infty(\mathbb{D},\mathbb{R}^3)$, we need to show that this limit limit is smooth, which is not possible since we only know that $\{E(\Phi_k)\}_{k\in\mathbb{N}}$ is bounded, that is, the gradient of the function is uniformly square integrable.

The basic principle of the calculus of variations is to find a solution in a class that is stable for weak convergence, and then to show the regularity of the limit (this is generally the most technical part of the proof, and we will not be able to say much about that in general). This approach is known under the name of the *direct method of the calculus of variations* that we have mentioned above. The right class of functions* is known under the name of Sobolev spaces ([2]). Those spaces appear for seemingly simple problems like harmonic maps with values into manifolds ([14, 15]). If $\Omega \subset \mathbb{R}^d$ is an open subset, for all $1 \leq p \leq \infty$, we have

$$W^{1,p}(\Omega) = L^p(\Omega) \cap \left\{ u : \begin{cases} \exists\, g_1, \cdots, g_d \in L^p(\Omega) \ \text{such that} \\ \displaystyle\int_\Omega u\, \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega g_i\, \varphi\, dx \ \ \forall \varphi \in C^\infty_c(\Omega) \ \ \forall 1 \leq i \leq d \end{cases} \right\}, \tag{1.2.4}$$

where $C^\infty_c(\Omega) \subset C^\infty(\Omega)$ is the space of smooth functions with compact support in Ω. In other words, we have $\varphi \in C^\infty_c(\Omega)$ if and only if $\varphi \in C^\infty(\Omega)$ and if there exits a compact subset $K \subset \Omega$ such that $\varphi = 0$ in $\Omega \setminus K$. The Sobolev space $W^{1,p}$ can be seen as a space of distributions (in the sense of Schwartz) that belong to L^p and whose weak derivatives also belong to L^p. For the problem of Plateau, we will consider a subspace of de $W^{1,2}(\mathbb{D})$ that has the right stability properties under weak convergence.

*For many problems, but it is often necessary to use more “exotic spaces” (Hardy spaces, BMO space (here, BMO stands for *Bounded Mean Oscillation*), Lorentz spaces, Besov spaces, etc. There is a whole zoo of spaces and one of the main tasks of the analyst is to find the “right”functional space for the considered problem, which is similar to the algebraic geometer who must find the right cohomology).

Chapter 2

Sobolev Spaces

2.1 A General Result

Let us start by an elementary result that is the prototype of theorem used in the calculus of variations.

Theorem 2.1.1. *Let $(X, \mathscr{T})$ be a topological space and let $f : X \to \mathbb{R} \cup \{\infty\}$ be a sequentially lower semi-continuous function, i.e., assume that for all $x \in X$ and for all sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that $x_n \underset{n\to\infty}{\longrightarrow} x$, we have*

$$F(x) \leq \liminf_{n\to\infty} F(x_n).$$

Furthermore, assume that F is coercive, i.e., that for all $t \in \mathbb{R}$, the level sets

$$F_t = X \cap \{x : F(x) \leq t\}$$

are sequentially pre-compact. *Then, F admits a minimiser.*

Proof. Assume without loss of generality that $F \neq \infty$. Then, there exists $x \in X$ such that $F(x) < \infty$, so we can pick a minimising sequence $\{x_n\}_{n\in\mathbb{N}}$, and since $\inf F(X) < \infty$, there exists $t \in \mathbb{R}$ such that $\{x_n\}_{n\in\mathbb{N}} \subset F_t$, which shows that $\{x_n\}_{n\in\mathbb{N}}$ admits a converging subsequence (let us write $x \in X$ its limit), that we still write $\{x_n\}_{n\in\mathbb{N}}$ for simplicity. By definition of a minimisation sequence, we have

$$F(x_n) \underset{n\to\infty}{\longrightarrow} \inf F(X).$$

On the other hand, the lower semi-continuity shows that

$$F(x) \leq \liminf_{n\to\infty} F(x_n) = \inf F(X).$$

Since $F(x) \geq \inf F(X)$, we finally deduce that x is a minimiser of F. $\square$

In this course, we will be mostly interested to minimise functionals of the form

$$L(u) = \int_\Omega F(x, \nabla u, \nabla^2 u, \cdots, \nabla^k u)dx,$$

where $\Omega \subset \mathbb{R}^d$ is a fixed open subset and $u : \Omega \to \mathbb{R}^n$ is a C^k function. For this, we will have to introduce Sobolev spaces, that are generalisations of Lebesgue spaces (L^p spaces, $1 \leq p \leq \infty$) and provide the suitable framework for the calculus of variations. However, before we delve into the existence theory, let us introduce the fundamental notions of the Euler-Lagrange equation. For this, we need elementary facts on compactly supported functions.

2.2 Compactly supported functions

Let Ω be an open set of $\mathbb{R}^d$, and define the space space of smooth functions of compact support by

$$\mathscr{D}(\Omega) = C_c^\infty(\Omega) = C^\infty(\Omega) \cap \{\varphi : \mathrm{supp}\,(\varphi) \subset\subset \Omega\}$$

However, we encounter a first difficulty: does there exist a non-zero function $\varphi \in C_c^\infty(\Omega)$? We will construct an important class of cut-off functions. First, introduce the function

$$\begin{aligned} f :&\mathbb{R} \to \mathbb{R}_+ \\ x &\mapsto \begin{cases} e^{-\frac{1}{x}} & \text{pour tout } \; x > 0 \\ 0 & \text{pour tout } \; x \leq 0. \end{cases} \end{aligned}$$

Let us show by induction that f belongs to $C^\infty(\mathbb{R})$. We have

$$e^{-\frac{1}{x}} \underset{x \to 0+}{\longrightarrow} 0,$$

which shows that $f \in C^0(\mathbb{R})$. For all $x > 0$, we have

$$\begin{aligned} f'(x) &= \frac{1}{x^2} e^{-\frac{1}{x}} \\ f''(x) &= \left(\frac{1}{x^4} - \frac{2}{x^3}\right) e^{-\frac{1}{x}}. \end{aligned}$$

By induction, let us show that for all $n \in \mathbb{N}$, there exists a polynomial of degree $2n$ such that

$$f^{(n)}(x) = P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x}}.$$

This is true for $n = 0$ and $n = 1$. If the property is verified for $f^{(n)}$, then

$$f^{(n+1)}(x) = \frac{d}{dx}\left(P_n\left(\frac{1}{x}\right)\right) e^{-\frac{1}{x}} = \left(\frac{1}{x^2} P_n\left(\frac{1}{x}\right) - \frac{1}{x^2} P_n'\left(\frac{1}{x}\right)\right) e^{-\frac{1}{x}} = P_{n+1}\left(\frac{1}{x}\right) e^{-\frac{1}{x}},$$

where

$$P_{n+1}(x) = x^2(P_n(x) - P_n'(x))$$

which is indeed a polynomial of degree $2n + 2 = 2(n+1)$. Finally, using the limit

$$\lim_{x \to 0+} \frac{1}{x^n} e^{-\frac{1}{x}} = \lim_{y \to \infty} \frac{y^n}{e^y} = 0,$$

that comes from the elementary comparison

$$e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!} \geq \frac{y^{n+1}}{(n+1)!},$$

we deduce that for all $n \in \mathbb{N}$,

$$\lim_{x \to 0+} f^{(n)}(x) = \lim_{x \to 0+} P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x}} = 0.$$

As a consequence, $f \in C^\infty(\mathbb{R})$, and $g(x) = f(|x|^2) f(1 - |x|^2) \in C_c^\infty(\mathbb{R}^d) \setminus \{0\}$. This allows us to construct for all $x \in \mathbb{R}^n$ and $r > 0$ a function $\varphi \in \mathscr{D}(B(x, 2r))$ such that $\varphi \geq 0$ and $\varphi = 1$ in $B(x, r)$.

The major interest in the calculus of variations is that those functions are *dense* in $L^1_{\mathrm{loc}}(\Omega)$.

Lemma 2.2.1. *Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and assume that for all $\varphi \in C_c^\infty(\Omega)$,*

$$\int_\Omega f(x)\varphi(x) = 0.$$

Then, $f = 0$ identically.

Proof. Let $K \subset \Omega$ be a compact set. Since f is measurable, the function $f_K = \text{sign}(f)\mathbf{1}_K$ is also measurable and belongs to $L^1(\Omega)$. Now, by a standard result on convolution, we find a sequence $\{\varphi_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\Omega)$ such that $\varphi_k \underset{k\to\infty}{\longrightarrow} f_K$ in $L^1(\Omega)$. Therefore, a direct application of the Theorem of Dominated Convergence shows that

$$0 = \lim_{k\to\infty} \int_\Omega f(x)\varphi_k(x)dx = \int_\Omega f(x)f_K(x)dx = \int_K |f(x)|dx.$$

Therefore, $f_K = 0$, and since the result holds for arbitrary compact K, we deduce that $f = 0$ identically. □

2.3 Euler-Lagrange Equation

Now, we let $\Omega \subset \mathbb{R}^d$ be a bounded open subset, $n \geq 1$, and $F \in C^2(\Omega, \mathbb{R}^n, M_{n,d}(\mathbb{R}))$. To distinguish partial derivatives, we write $F = F(x, \xi, P)$. We introduce the energy

$$E(u) = \int_\Omega F(x, u, \nabla u)dx.$$

Theorem 2.3.1. *Assume that $u \in C^2(\Omega, \mathbb{R}^n) \cap C^0(\overline{\Omega}, \mathbb{R}^n)$ minimises E amongst all functions $v \in C^2(\Omega, \mathbb{R}^n) \cap C^0(\overline{\Omega}, \mathbb{R}^n)$ such that $v = u$ on $\partial\Omega$. Then, the following equation is verified*

$$\text{div}\left(\nabla_P F(x, u, \nabla u)\right) = \nabla_\xi F(x, u, \nabla u).$$

Proof. The proof follows the one from the introduction and we omit it. □

Definition 2.3.2. The equation is called the Euler-Lagrange equation.

The main difficulty is to give sense to the Euler-Lagrange equation for non-smooth functions, and this will force us to introduce distributions and Sobolev functions. Before doing so, let us give a few examples.

Example 2.3.3.
1. If $F(x, \xi, P) = \frac{1}{2}|P|^2$, then the associated Euler-Lagrange equation is the Laplace equation

$$\Delta u = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} = 0.$$

2. If $F(x, \xi, X) = \frac{1}{p}|X|^p$, then we get the so-called p-harmonic maps:

$$\text{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0.$$

Notice that if $1 < p < 2$, then the equation is degenerate.

3. We have already mentioned the equation of minimal surfaces: $F(x, \xi, P) = \sqrt{1 + |P|^2}$. Its Euler-Lagrange equation is given by

$$\text{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0,$$

or alternatively

$$(1 + |\nabla u|^2)\Delta u - (\nabla u)^t \cdot \nabla^2 u \cdot (\nabla u) = 0.$$

4. If $d = m = 2$, and $F(x, \xi, P) = \dfrac{1}{2}|P|^2 + f(\det(\xi))$, where $f : \mathbb{R} \to \mathbb{R}$ is a real-valued function, one checks that the Euler-Lagrange equation is given by

$$\begin{cases} \Delta u_1 + \dfrac{\partial}{\partial x_1}\left(f'(\det \nabla u)\dfrac{\partial u_2}{\partial x_2}\right) - \dfrac{\partial}{\partial x_2}\left(f'(\det \nabla u)\dfrac{\partial u_2}{\partial x_1}\right) = 0 \\ \Delta u_2 + \dfrac{\partial}{\partial x_1}\left(f'(\det \nabla u)\dfrac{\partial u_1}{\partial x_2}\right) + \dfrac{\partial}{\partial x_2}\left(f'(\det \nabla u)\dfrac{\partial u_1}{\partial x_1}\right) = 0 \end{cases}$$

2.4 Basic Notions on Distributions

Let us first introduce a topology on the set of compactly supported functions.

For all compact $K \subset \Omega$, we define

$$\mathscr{D}_K(\Omega) = C^\infty(\Omega) \cap \{\varphi : \mathrm{supp}(\varphi) \subset K\}.$$

Notice that

$$\mathscr{D}(\Omega) = \bigcup_{\substack{K \subset \Omega \\ K \text{ compact}}} \mathscr{D}_K(\Omega)$$

For all $m \in \mathbb{N}$ and compact subset $K \subset \mathscr{D}(\Omega)$, define the semi-norm on $\mathscr{D}(\Omega)$ by

$$\|\varphi\|_{m,K} = \sup_{|\alpha| \leq m} \|D^\alpha \varphi\|_{\mathrm{L}^\infty(K)}.$$

If $\{K_n\}_{n\in\mathbb{N}}$ is an exhaustive sequence of compact sets of Ω, the vector space $(\mathscr{D}(\Omega), \{\|\cdot\|_{m\in\mathbb{N}}\})$ can be equipped with a distance:

$$d(\varphi, \psi) = \sum_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} \frac{1}{2^{m+n}} \frac{\|\varphi - \psi\|_{m,K_n}}{1 + \|\varphi - \psi\|_{m,K_n}},$$

We will give an *ad hoc* definition for distributions, and see that in all reasonable cases (L^p functions, Radon measures, etc), the objects that we will consider are distributions.

In the case $d = 1$, if $I =]a, b[\subset \mathbb{R}$ is an open interval, intuitively, a distribution is a *linear* map $T : \mathscr{D}(I) \to \mathbb{R}$ such that for some locally integrable functions $f_0, \cdots, f_m \in L^1_{\mathrm{loc}}(I)$, we have for all $\varphi \in \mathscr{D}(I)$

$$T(\varphi) = \sum_{k=0}^{m} \int_I f_k(x) \varphi^{(k)}(x) dx, \tag{2.4.1}$$

where $\varphi^{(0)}(x) = \varphi(x)$ and for all $k \geq 1$

$$\varphi^{(k)}(x) = \frac{d^k}{dx^k}\varphi(x).$$

All distributions that will appear in the lecture will take this form, but it is not always obvious to see that a distribution reduces to this form. Let us now give the formal definition of distributions that follows the above intuitive characterisation (2.4.1).

Definition 2.4.1. Let $\Omega \subset \mathbb{R}^d$ be an open set. We say that a linear map $T : \mathscr{D}(\Omega) \to \mathbb{R}$ is a distribution if for all *compact* subset $K \subset \Omega$, there exists a constant $C_K < \infty$ and an integer $m_K \in \mathbb{N}$ such that for all $\varphi \in \mathscr{D}_K(\Omega)$

$$|T(\varphi)| \leq C_K \|\varphi\|_{m,K} = C_K \sup_{|\alpha| \leq m_K} \sup_{x \in K} |D^\alpha \varphi(x)|, \tag{2.4.2}$$

where we denote $|\alpha| = \alpha_1 + \cdots + \alpha_d$ for all $\alpha \in \mathbb{N}^d$. We will also use the notation

$$\langle T, \varphi\rangle = T(\varphi)$$

that is sometimes a more convenient notation to show the bilinearity in T and φ of the pairing.

The smallest integer $m_K \in \mathbb{N}$ such that the inequality (2.4.2) holds true (for some constant $C_K < \infty$) is called the *order* of the distribution and we denote it

$$\mathrm{ord}_K(T) \in \mathbb{N}.$$

The order of T is given by

$$m = \sup_{\substack{K\subset\Omega\\ K \text{ compact}}} \mathrm{ord}_K(T) \in \mathbb{N}\cup\{\infty\}\,.$$

We say that T is a distribution of finite order if $\mathrm{ord}(T) < \infty$.

Remark 2.4.2. Analogously, we define complex-valued, or vector-valued definition by taking the product spaces of distributions.

We will not define the general topology of distributions and stick to the one for sequences, which will suffice for our purpose.

Definition 2.4.3. We say that sequence $\{T_n\}_{n\in\mathbb{N}} \subset \mathscr{D}'(\Omega)$ converges to an element $T \in \mathscr{D}'(\Omega)$ if and only if

$$T_n(\varphi) \underset{n\rightarrow\infty}{\longrightarrow} T(\varphi) \quad \text{for all } \varphi \in \mathscr{D}(\Omega).$$

Remark 2.4.4. Analogously, we define complex-valued, or vector-valued definition by taking the product spaces of distributions. Notice that distributions of order 0 are Radon measures. We will see examples below

Examples 2.4.5. **1**. If $f \in L^1_{\mathrm{loc}}(\Omega)$, then the distribution $T = f$ defined by integration such that

$$T(\varphi) = \int_\Omega f\,\varphi dx \quad \text{for all } \varphi \in \mathscr{D}(\Omega)$$

is a distribution of order 0, with $\|T\|_K = \|f\|_{\mathrm{L}^1(K)}$. More generally, if $T = \mu$ is a real Radon measure, then

$$T(\varphi) = \int_\Omega \varphi\, d\mu$$

is also a distribution of order 0, such that $\|T\|_K = \mu(K)$. An important example is the Dirac mass at $x_0 \in \Omega$, given by

$$\delta_{x_0}(\varphi) = \varphi(x_0).$$

2. The Dirac mass δ_a such that $\delta_a(\varphi) = \varphi(a)$ $(a \in \Omega)$ is a very important distribution (a measure, in fact), that will have a crucial importance in several theorems for reasons that will be made clear by convolution and Fourier transform.

3. Anticipating on the next section, for all $a \in \mathbb{R}$ define for all $n \in \mathbb{N}$ the distribution $\delta_a^{(n)} \in \mathscr{D}'(\mathbb{R})$ by $\delta_a^{(n)}(\varphi) = (-1)^n\varphi^{(n)}(a)$. Then, the following distribution

$$T = \sum_{n\in\mathbb{N}} \delta_n^{(n)}$$

has infinite order. Indeed, we see easily that for all $n \in \mathbb{N}$, the restriction of T to $B(0, n + \frac{1}{2})$ has order n.

3. Let $\Gamma \subset \mathbb{R}^d$ be a C^1 curve. Then, we have

$$|\Gamma|(\varphi) = \int_{\Gamma} \varphi \, dl \in \mathscr{D}'(\mathbb{R}^d).$$

Indeed, if $\gamma : [0,1] \to \mathbb{R}^d$ is a parametrisation of Γ, we have

$$||\Gamma|(\varphi)| = \left| \int_0^1 \varphi(\gamma(t))|\gamma'(t)|dt \right| \leq \sup_{x\in\mathbb{R}} |\varphi(x)| \int_0^1 |\gamma'(t)|dt = \mathscr{L}(\Gamma) \sup_{x\in\mathbb{R}^d} |\varphi(x)| < \infty,$$

which shows that $|\Gamma|$ is a distribution. Likewise, if $\Sigma \subset \mathbb{R}^3$ is a C^1 surface, define $|\Sigma| \in \mathscr{D}'(\mathbb{R}^3)$ by

$$|\Sigma|(\varphi) = \int_{\Sigma} \varphi \, dA.$$

If $f : \Omega \to \mathbb{R}^3$ is a parametrisation of Σ (where $\Omega \subset \mathbb{R}^2$ is open), then

$$||\Sigma|(\varphi)| = \left| \int_{\Omega} \varphi(f(x,y))|\partial_x f \times \partial_y f| dx \, dy \right| \leq \text{Area}(\Sigma) \sup_{z\in\mathbb{R}^3} |\varphi(z)| < \infty.$$

4. Likewise, if $\Gamma \subset \mathbb{R}^d$ is a C^1 curve, define for all smooth vector field $X : \mathbb{R}^d \to \mathbb{R}^d$

$$[\Gamma](X) = \int_{\Gamma} X \cdot dl.$$

Arguing similarly, one shows that $[\Gamma] \in \mathscr{D}'(\mathbb{R}^d)$. Assume now that $d = 3$. If Σ is a C^1 surface, define for all $X \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$

$$[\Sigma](X) = \int_{\Sigma} X \cdot dA.$$

It is easy to see that $[\Sigma] \in \mathscr{D}'(\mathbb{R}^3, \mathbb{R}^3)$. Assuming that $\partial\Sigma$ is a C^1 curve, Stokes theorem shows that

$$[\Sigma](\text{rot}\, X) = \int_{\Sigma} \text{rot}\, X \cdot dA = \int_{\partial\Sigma} X \cdot dl = [\partial\Sigma](X),$$

which allows one to elegantly rephrase Stokes theorem in this particular case ($[M](d\omega) = [\partial M](\omega)$). In fact, this formula permits to generalise Stokes formula to non-smooth surfaces, and forms the basis of the theory of Federer-Fleming ([11, Chapter 4]).

5. The principal value integral (at 0) of a function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that $f \in L^1_{\text{loc}}(\mathbb{R} \setminus \{0\})$ is defined by

$$\text{p.v.} f(\varphi) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} f(x)\varphi(x)dx.$$

Under suitable conditions on f, p.v.f is a well-defined distribution, known as Cauchy principal value. Take $f(x) = \frac{1}{x}$. Then, by oddness of f, for all $0 < \varepsilon < R < \infty$, we have

$$\left\langle \text{p.v.}\frac{1}{x}, f \right\rangle = \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \varphi(x)\frac{dx}{x} = \int_{[-R,-\varepsilon]\cup[\varepsilon,R]} (\varphi(x) - \varphi(0))\frac{dx}{x} + \int_{\mathbb{R}\setminus[-R,R]} \varphi(x)dx,$$

and since φ is of class C^1, the function $\frac{\varphi(x)-\varphi(0)}{x}$ is bounded at 0, and for all $0 < R < \infty$, we have

$$\left\langle \text{p.v.}\frac{1}{x}, f \right\rangle = \int_{-\infty}^{-R} \varphi(x)\frac{dx}{x} + \int_{-R}^{R} (\varphi(x) - \varphi(0))\frac{dx}{x} + \int_{R}^{\infty} \varphi(x)\frac{dx}{x}.$$

Taking $R > 0$ large enough such that $\text{supp}(\varphi) \subset [-R, R]$, we deduce by Fubini's theorem that

$$\left\langle \text{p.v.}\frac{1}{x}, \varphi \right\rangle = \int_{-R}^{R} (\varphi(x) - \varphi(0))\frac{dx}{x}$$

$$= -\int_{-R}^{0}\left(\int_{x}^{0}\varphi'(t)dt\right)\frac{dx}{x} + \int_{0}^{R}\left(\int_{0}^{x}\varphi'(t)dt\right)\frac{dx}{x}. \tag{2.4.3}$$

We first compute

$$\begin{aligned}\int_{-R}^{0}\left(\int_{x}^{0}\varphi'(t)dt\right)\frac{dx}{x} &= \int_{-R}^{0}\left(\int_{-R}^{0}\varphi'(t)\mathbf{1}_{\{x\leq t\leq 0\}}dt\right)\frac{dx}{x} = \int_{-R}^{0}\varphi'(t)\left(\int_{-R}^{t}\frac{dx}{x}\right)dt\\ &= \int_{-R}^{0}\varphi'(t)\log\left(\frac{t}{-R}\right)dt. \end{aligned}\tag{2.4.4}$$

Likewise, we have

$$\int_{0}^{R}\left(\int_{0}^{x}\varphi'(t)dt\right)\frac{dx}{x} = \int_{0}^{R}\varphi'(t)\log\left(\frac{R}{t}\right)dt, \tag{2.4.5}$$

which implies that

$$\begin{aligned}\left\langle \mathrm{p.v.}\frac{1}{x},\varphi\right\rangle &= -\int_{-R}^{R}\varphi'(t)\log\left(\frac{|t|}{R}\right)dt = \int_{-R}^{R}\varphi'(t)\log|t|dt + \log(R)\int_{-R}^{R}\varphi'(t)dt\\ &= -\int_{\mathbb{R}}\varphi'(t)\log|t|dt,\end{aligned}\tag{2.4.6}$$

since $\mathrm{supp}(\varphi)\subset[-R,R]$. This expression easily shows that $\mathrm{p.v.}\frac{1}{x}$ has order 1, and that the distributional derivative (as defined in the next section) of $x\mapsto\log|x|$ is $\mathrm{p.v.}\frac{1}{x}$. Indeed, for all $\varphi\in\mathscr{D}(\mathbb{R})$, we have

$$\left|\int_{\mathbb{R}}\varphi'(t)\log|t|dt\right| \leq \left(\int_{\mathrm{supp}(\varphi')}\log|t|dt\right)\|\varphi'\|_{\mathrm{L}^\infty(\mathbb{R})},$$

which shows that $\mathrm{p.v.}\frac{1}{x}$ has order at most 1. If this distribution had order 0, it would extend to a Radon measure. We will therefore exhibit a bounded sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ in $C_c(\mathbb{R})$ such that $\left\langle\mathrm{p.v.}\frac{1}{x},\varphi_n\right\rangle$ diverges. Now, let $\{\varphi_n\}_{n\in\mathbb{N}}\subset C^0_c(\mathbb{R})$ such that φ_n is odd, $\mathrm{supp}(\varphi_n)\subset[-2,2]$, $\varphi_n=\varphi_0$ on $[-2,2]\setminus[-1,1]$,

$$\begin{cases}\varphi_n(x)=-1 & \text{for all } -1\leq x\leq -\dfrac{1}{n}\\ \varphi_n(x)=nx & \text{for all } -\dfrac{1}{n}\leq x\leq\dfrac{1}{n}\\ \varphi_n(x)=1 & \text{for all } \dfrac{1}{n}\leq x\leq 1.\end{cases}$$

This sequence is bounded in $C_c(\mathbb{R})$ since $\mathrm{supp}(\varphi_n)\subset[-2,2]$, and $|\varphi_n|\leq 1$. However, we have

$$\begin{aligned}\left\langle\mathrm{p.v.}\frac{1}{x},\varphi_n\right\rangle &= \lim_{\varepsilon\to 0}\left(2\int_{1}^{2}\varphi_0(x)\frac{dx}{x} + 2\int_{\frac{1}{n}}^{1}\frac{dx}{|x|} + 2\int_{\varepsilon}^{\frac{1}{n}}n\,dx\right)\\ &= 2\int_{1}^{2}\varphi_0(x)\frac{dx}{x} + 2 + 2\log(n)\underset{n\to\infty}{\longrightarrow}\infty.\end{aligned}$$

Therefore, we deduce that $\mathrm{p.v.}\frac{1}{x}$ is a distribution of order exactly 1. By introducing $\varphi(x)-\varphi(-x)$, give an alternative proof of the above results.

The first basic property of distributions is the multiplication by smooth functions. Recall that $\mathscr{E}(\Omega)=C^\infty(\Omega)$ equipped with the compact-open topology (which makes it a Fréchet space).

Definition 2.4.6. For all $T\in\mathscr{D}'(\Omega)$ and $f\in C^\infty(\Omega)$, the product $S=f\,T$ defined by

$$\langle f\,T,\varphi\rangle = \langle T, f\,\varphi\rangle \quad \text{for all } \varphi\in\mathscr{D}(\Omega)$$

is a distribution such that for all compact $K\subset\Omega$, we have

$$\mathrm{ord}_K(f\,T)\leq\mathrm{ord}_K(\varphi). \tag{2.4.7}$$

Remark 2.4.7. That $f\,T \in \mathscr{D}'(\Omega)$ follows immediately since $f\varphi \in \mathscr{D}(\Omega)$ for all $(f,\varphi) \in C^\infty(\Omega)\times\mathscr{D}(\Omega)$, and the property of order is trivial by Leibniz formula.

In general, the product of two distributions makes no sense, but this is expected since even in the case of L^p functions, we know that if $f \in L^p(\Omega)$, then $fg \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ if and only if $g \in L^{p'}_{\mathrm{loc}}(\mathrm{supp}(f))$, where $\dfrac{1}{p}+\dfrac{1}{p'}=1$.

Example 2.4.8. We have $x\cdot\left(\mathrm{p.v.}\frac{1}{x}\right)=1$. Indeed, for all $\varphi\in\mathscr{D}(\Omega)$,

$$\left\langle x\cdot\left(\mathrm{p.v.}\frac{1}{x}\right),\varphi\right\rangle=\left\langle \mathrm{p.v.}\frac{1}{x},x\,\varphi\right\rangle=\lim_{\varepsilon\to0}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}\Big(x\,\varphi(x)\Big)\frac{dx}{x}=\int_{\mathbb{R}}\varphi(x)dx=\langle 1,\varphi\rangle\,.$$

We saw above that $\mathrm{p.v.}\frac{1}{x}$ was a derivative of a Radon measure (of a function, more precisely). This fact is general to distributions of finite support, but we will not prove this result.

The previous example also poses the question of division of distributions, but we will not address it in those lecture notes.

2.4.1 Differentiation of Distributions

The fundamental idea of Schwartz (1945) is to show that by duality, one can define differentiation of distributions, and that this operation is continuous with respective to either topology—weak or strong—on $\mathscr{D}'(\Omega)$.

Definition 2.4.9. For all multi-index $\alpha\in\mathbb{R}^d$, and $T\in\mathscr{D}'(\Omega)$, we define $D^\alpha T\in\mathscr{D}'(\Omega)$ to be the distribution satisfying

$$D^\alpha T(\varphi)=(-1)^{|\alpha|}T(D^\alpha\varphi).$$

It satisfies $\mathrm{ord}_K(D^\alpha T)\leq \mathrm{ord}_K(T)+|\alpha|$ for all compact $K\subset\Omega$.

The continuity of this operation for the weak topology is trivial for

$$|T(D^\alpha\varphi)|\leq\|T\|_K\,\|D^\alpha\varphi\|_{m,K}\leq\|\varphi\|_{m+|\alpha|,K}$$

holds for all compact subset $K\subset\Omega$.

In early work, Schwartz had not introduced the minus sign ([20]), but the sign convention is the one consistent with integration by parts.

Of course, if $T=f\in C^\infty(\Omega)$, integrating by parts, we deduce that

$$\left\langle\frac{\partial}{\partial x_i}T,\varphi\right\rangle=-\left\langle T,\frac{\partial}{\partial x_i}\varphi\right\rangle=-\int_\Omega f\,\partial_{x_i}\varphi\,dx=\int_\Omega\varphi\,\partial_{x_i}f\,dx,$$

so that $\partial_{x_i}T=\partial_{x_i}f$. Sobolev spaces, which will make for half of those lectures, are sets of distributions whose weak derivatives belong to some L^p space (this will be treated in the Sobolev inequality below). Thinking about partial differential equation (energy functionals), it becomes apparent why Sobolev spaces are the natural settings to solve equations, and their good properties allows one to use (say) calculus of variation in order to build solutions.

Examples 2.4.10. Let $H=\mathbf{1}_{\mathbb{R}_+}$ be the Heaviside function. Then, we have $H'=\delta_0$ in the sense of distributions. Indeed, for all $\varphi\in\mathscr{D}(\mathbb{R})$, we have

$$\langle H',\varphi\rangle=-\langle H,\varphi'\rangle=-\int_{\mathbb{R}}H(x)\varphi'(x)dx=-\int_0^\infty\varphi'(x)dx=\varphi(0)=\langle\delta_0,\varphi\rangle\,.$$

We saw in the first example that for C^1 functions by arcs, the usual derivative and the distributional derivative coincide up to Dirac masses. This is a general fact, the formula of "jumps" allows on to quantify the difference (both quantities only differ up to Dirac masses).

Theorem 2.4.11. *Let $I \subset \mathbb{R}$ an open interval, and $f : I \to \mathbb{R}$ be a C^1 function by arcs,* i.e. *a function such that there exists $\inf I < a_1 < \cdots < a_n < \sup I$ such that $f|_{(a_i,a_{i+1})}$, $f_{(\inf I, a_1)}$ and $f_{(a_n, \sup I)}$ are functions of class C^1. Then, we have*

$$f' = \sum_{i=1}^{n-1} f'\mathbf{1}_{(a_i,a_{i+1})} + f'\mathbf{1}_{(\inf I,a_1)} + f'\mathbf{1}_{(a_n,\sup I)} + \sum_{i=1}^{n} \left(f'(a_i^+) - f'(a_i^-)\right)\delta_{a_i},$$

where $f'(a_i^{\pm}) = \lim\limits_{x \to a_i^{\pm}} f'(x)$ for all $i \in \{1, \cdots, n\}$.

Proof. The proof goes exactly as in Examples 2.4.10 and is omitted. □

Remark 2.4.12. This formula has generalisations to higher dimension, but would force us to introduce notions of differential geometry, that we consider to be outside the scope of those lectures.

The basic theorem about differentiation shows that the solution to an elliptic equation is generally unique in $\mathscr{D}'(\mathbb{R}^d)$. There are deep theorems that involve Sobolev spaces—to be introduced in the next chapter—and we will simply mention elementary results related to continuous functions and first order derivatives.

Theorem 2.4.13. *Let $T \in \mathscr{D}'(\mathbb{R}^d)$ be such that $\nabla T = 0$. Then, there exists $\vec{C}_0 \in \mathbb{C}^n$ such that $T = \vec{C}_0$.*

Proof. Showing by induction that $\partial_{x_i} T = 0$ implies that T is independent of x_i,we need only show the result for $d = 1$. Assume that $T' = 0$, and separating real and complex part, assume without loss of generality that T is real-valued. For all $\varphi \in \mathscr{D}(\mathbb{R})$, we have $\varphi = \psi'$ for some $\psi \in \mathscr{D}(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} \varphi(x)dx = 0. \tag{2.4.8}$$

Denote by H the hyperplane of such functions. Indeed, provided that (2.4.8) holds, we deduce that the following function

$$\psi(x) = \int_{-\infty}^{x} \varphi(y)dy$$

belongs to $\mathscr{D}(\mathbb{R})$ since φ has compact support. Conversely, if $\varphi = \psi'$ with $\psi \in \mathscr{D}(\mathbb{R})$, we have

$$\int_{-\infty}^{x} \varphi(y)dy = \psi(x)$$

And since ψ has compact support, there exists $r \in \mathbb{R}$ such that $\psi(x) = 0$ for all $x \geq r$, which shows that (2.4.8) holds in particular. Now, let $\theta \in \mathscr{D}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \theta(x)dx = 1.$$

For all $\varphi \in \mathscr{D}(\mathbb{R})$, we have

$$\psi = \varphi - \theta \int_{\mathbb{R}} \varphi \, d\mathscr{L}^1 \in H,$$

which implies since $T' = 0$ that

$$0 = T(\psi) = T(\varphi) - T(\theta) \int_{\mathbb{R}} \varphi \, d\mathscr{L}^1,$$

or

$$T = c_0 = T(\theta).$$

This concludes the proof of the theorem. □

2.5 Definition of Sobolev Spaces and Basic Properties

Let Ω be an open set of $\mathbb{R}^d$ for some $d \geq 1$, that we choose connected for convenience.

Definition 2.5.1. Let $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. A function $u \in L^1_{\mathrm{loc}}(\Omega)$ belongs to the Sobolev space $W^{m,p}(\Omega)$ if and only if for all $|\alpha| \leq m$, we have $D^\alpha u \in L^p(\Omega)$, where D^α is its distributional derivative. In other words, for all $|\alpha| \leq m$, there exists $f_\alpha \in L^p(\Omega)$ such that for all $\varphi \in C^\infty_c(\Omega)$, we have

$$\int_\Omega u\, D^\alpha \varphi\, dx = (-1)^{|\alpha|} \int_\Omega f_\alpha\, \varphi\, dx, \tag{2.5.1}$$

and we write $D^\alpha u = f_\alpha$.

If $p = 2$, we commonly write $W^{m,2}(\Omega) = H^m(\Omega)$. We equip $W^{m,p}(\Omega)$ with the following norm

$$\|u\|_{\mathrm{W}^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{\mathrm{L}^p(\Omega)}. \tag{2.5.2}$$

Remark 2.5.2. If we forget about distribution theory, we can therefore take (2.5.1) as a definition. But distributions are still useful, since for example, the dual of a Sobolev space is a space of distributions.

Theorem 2.5.3. *The space $W^{m,p}(\Omega)$ is a Banach space. The space $W^{m,p}(\Omega)$ is reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$. The space $H^m(\Omega)$ is a separable Hilbert space.*

Proof. **Step 1.** $W^{m,p}$ is a Banach space.

Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{m,p}(\Omega)$ be a Cauchy sequence. Since $L^p(\Omega)$ is a Banach space, there exists $u \in L^p(\Omega)$ and for all $0 < |\alpha| \leq m$, there exists $u_\alpha \in L^p(\Omega)$ such that $u_n \underset{n\to\infty}{\longrightarrow} u$ and $D^\alpha u_n \underset{n\to\infty}{\longrightarrow} u_\alpha$. Now, by Hölder's inequality, for all $\varphi \in \mathscr{D}(\Omega)$, we have

$$|\langle u_n, \varphi\rangle - \langle u, \varphi\rangle| \leq \|u_n - u\|_{\mathrm{L}^p(\Omega)} \|\varphi\|_{\mathrm{L}^{p'}(\Omega)} \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore, $u_n \underset{n\to\infty}{\longrightarrow} u$ in the distributional sense, and since derivation is continuous under $\sigma(\mathscr{D}(\Omega), \mathscr{D}'(\Omega))$, we deduce that $u_\alpha = D^\alpha u$ for all $|\alpha| \leq m$, which concludes the proof.

Step 2. Other properties.

We have an isometry $W^{m,p}(\Omega) \to L^p(\Omega)^{N(d,m)}$ given by the natural map $u \mapsto \{D^\alpha u\}_{|\alpha|\leq m}$, where

$$N(d, m) = \mathrm{card}(\mathbb{N}^d \cap \{\alpha : |\alpha| \leq m\}).$$

In particular, $W^{m,p}(\Omega)$ is a closed space of $L^p(\Omega)^{N(d,m)}$, which implies the claims on reflexivity and separability. □

Remark 2.5.4. There are many generalisations of Sobolev spaces, using more complicated norms or weaker notions than functions. We will not list them all, but let us nevertheless mention the important class of function of *bounded variations*, commonly called BV functions, that are L^1 functions whose distributional derivative is a Radon measure. Those functions have applications to the study of minimal surfaces, and we send to Giusti's monograph for more details ([12]).

Theorem 2.5.5. *Let $u \in W^{m,p}(\Omega)$, with $1 \leq p < \infty$. Then, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(\mathbb{R}^d)$ such that*

$$\begin{cases} \|u_n - u\|_{\mathrm{L}^p(\Omega)} \underset{n\to\infty}{\longrightarrow} 0 & \\ \|D^\alpha(u_n - u)\|_{\mathrm{L}^p(\Omega')} \underset{n\to\infty}{\longrightarrow} 0 & \textit{for all } \Omega' \subset\subset \Omega. \end{cases} \tag{2.5.3}$$

Proof. Let $\{\rho_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(\mathbb{R}^d)$ be an approximation of unity, *i.e.* a non-negative function with integral 1, support included in $B(0, \frac{1}{n})$, and such that $\rho_n \underset{n\to\infty}{\longrightarrow} \delta_0$ in $\mathscr{D}'(\mathbb{R}^d)$. Let $v_n = \rho_n * (u\,\mathbf{1}_\Omega)$. Then, the classical results of convolution show that

$$\|u_n - u\,\mathbf{1}_\Omega\|_{\mathrm{L}^p(\Omega)} \underset{n\to\infty}{\longrightarrow} 0,$$

which shows the first part of (2.5.3). Now, fix some relatively compact open subset Ω' of Ω, and let $\chi \in \mathscr{D}(\Omega)$ such that $\chi = 1$ on an open neighbourhood of Ω'. Then, for $n \in \mathbb{N}$ large enough, we have

$$\rho_n * (\chi\, u) = \rho_n * (u\, \mathbf{1}_\Omega).$$

Indeed, we have

$$\mathrm{supp}\,(\rho_n * (\chi\, u) - \rho_n * (u\, \mathbf{1}_\Omega)) = \mathrm{supp}(\rho_n * ((1_\Omega - \chi)u)) \subset \mathrm{supp}(\rho_n) + \mathrm{supp}(1_\Omega - \chi) \subset \Omega \setminus \overline{\Omega'}$$

for $n \in \mathbb{N}$ large enough. Indeed, $\mathrm{supp}\,(1_\Omega - \chi) \subset \Omega \setminus \overline{\Omega'}$ which is an open set, and since $\mathrm{supp}\,(\rho_n) \subset B(0, \frac{1}{n})$, for n large enough, we also have

$$B\left(0, \frac{1}{n}\right) + \mathrm{supp}(1_\Omega - \chi) \subset \Omega \setminus \overline{\Omega'}.$$

Now, we have

$$D^\alpha(\rho_n * (\chi u)) = \rho_n * (u\, D^\alpha \chi + \chi\, D^\alpha u) \qquad \text{in } \mathscr{D}'(\mathbb{R}^d).$$

In particular, we have

$$\|D^\alpha(v_n - u)\|_{\mathrm{L}^p(\Omega')} \underset{n\to\infty}{\longrightarrow} 0.$$

Finally, if $\eta \in C^\infty(\Omega)$ is such that $\eta(t) = 1$ for $t \leq 1$ and $\eta(t) = 0$ for $t \geq 2$, defining $\eta_n(x) = \eta\left(\frac{|x|}{n}\right)$, the sequence $\{u_n = \eta_n\, v_n\}_{n\in\mathbb{N}}$ has the required properties. ☐

Remark 2.5.6. More generally, the Meyers-Serrin theorem shows that for all $u \in W^{m,p}(\Omega)$, there exists $\{u_n\}_{n\in\mathbb{N}} \subset W^{m,p}(\Omega) \cap C^\infty(\Omega)$ such that $u_n \underset{n\to\infty}{\longrightarrow} u$ in $W^{m,p}(\Omega)$.

2.6 Sobolev Embedding Theorem

2.6.1 Super-Critical Case

As we mentioned previously, the Sobolev inequality shows that a distribution u such that $\nabla u \in L^p(\mathbb{R}^d)$ is in fact a locally L^q function for some exponent $q > 1$. Assuming that u belongs to some L^r space, we get a global estimate. In particular, the Sobolev inequality is particularly easy to state for $W^{1,p}$ functions. The argument generalises to $W^{m,p}$ spaces, and once more, we need only look at the case $m = 1$ to deduce more general Sobolev inequalities. The results depend on the relation between $1 \leq p \leq \infty$ and the ambient dimension d.

Theorem 2.6.1 (Sobolev)**.** *Assume that $d \geq 2$, and let $1 \leq p < d$. Then, we have a continuous embedding $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^{p^*}(\mathbb{R}^d)$, where*

$$p^* = \frac{dp}{d-p}.$$

For $d = 1$, for all interval $I \subset \mathbb{R}$, we have a continuous embedding $W^{1,p}(I) \hookrightarrow C^0(I)$, and

$$\|u\|_{\mathrm{L}^\infty(I)} \leq C(I)\, \|u\|_{\mathrm{W}^{1,p}(I)}\,.$$

Proof. We only treat the case $d = 1$ for simplicity.

Lemma 2.6.2. *Let $g \in L^1_{\mathrm{loc}}(I)$, and fix some $x_0 \in I$. Define*

$$f(x) = \int_{x_0}^x g(y)dy.$$

Then, we have $f \in C^0(I)$, and $f' = g$ in $\mathscr{D}'(I)$.

Proof. The continuity follows from the classical theorems of continuous dependence of the Lebesgue integral (one can use the dominated convergence theorem for example). Now, for all $\varphi \in \mathscr{D}'(I)$, we have by Fubini's theorem

$$\begin{aligned}
&\int_I f(x)\,\varphi'(x)\,dz = \int_I \left(\int_{x_0}^x g(y)dy\right)\varphi'(x)dx \\
&= -\int_{\inf I}^{x_0}\int_I g(y)\varphi'(x)\mathbf{1}_{\{x\leq y\leq x_0\}}dxdy + \int_{x_0}^{\sup I}\int_I g(y)\varphi'(x)\mathbf{1}_{\{x_0\leq y\leq x\}}dxdy \\
&= -\int_I\left(\int_{\inf I}^{x_0}\varphi'(x)\mathbf{1}_{\{x\leq y\leq x_0\}}dx\right)g(y)dy + \int_I\left(\int_{x_0}^{\sup I}\varphi'(x)\mathbf{1}_{\{x_0\leq y\leq x\}}dx\right)g(y)dy \\
&= -\int_I\left(\int_{\inf I}^{y}\varphi'(x)dx\right)g(y)dy + \int_I\left(\int_{y}^{\sup I}\varphi'(x)\right)g(y)dy = -\int_I \varphi(y)g(y)dy,
\end{aligned}$$

where we used that φ has compact support in I. Therefore, we have in the distributional sense $f' = g$ in $\mathscr{D}'(I)$ as claimed. □

Thanks to the lemma and Theorem 2.4.13, we deduce that for all $u \in W^{1,p}(I)$ and for all $x_0 \in I$, we have

$$u(x) - u(x_0) = \int_{x_0}^x u'(y)dy.$$

Provided that $I = \mathbb{R}$ and $u \in \mathscr{D}(\mathbb{R})$, we obtain similarly the formula

$$u(x)|u(x)|^{p-1} = \int_{-\infty}^x pu'(x)|u(x)|^{p-1}dx,$$

so that by Hölder's inequality

$$|u(x)|^p \leq p\,\|u'\|_{\mathrm{L}^p(\mathbb{R})}\,\|u\|_{\mathrm{L}^p(\mathbb{R})}^{p-1},$$

so that

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R})} \leq p^{\frac{1}{p}}\,\|u\|_{\mathrm{W}^{1,p}(\mathbb{R})}.$$

The general result follows by density of $\mathscr{D}(\mathbb{R})$ in $W^{1,p}(\mathbb{R})$. □

Recall the following elementary interpolation result.

Lemma 2.6.3. *Let (X,μ) be a measured space, $1 \leq p < q \leq \infty$, and $u \in L^p \cap L^q(X,\mu)$. Then, $u \in L^r(X,\mu)$ for all $p \leq r \leq q$, and we have*

$$\|u\|_{\mathrm{L}^r(X)} \leq \|u\|_{\mathrm{L}^p(X)}^{\alpha}\,\|u\|_{\mathrm{L}^q(X)}^{1-\alpha}, \tag{2.6.1}$$

where $\alpha \in [0,1]$ is such that

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q} \tag{2.6.2}$$

Proof. Let $p < r < q$ and $0 < \alpha < 1$ such that

$$r = \alpha p + (1-\alpha)q.$$

By the Hölder's inequality, for all $1 < s < \infty$, we have

$$\int_X |u|^r d\mu = \int_X |u|^{\alpha p}|u|^{(1-\alpha)q}d\mu \leq \left(\int_X |u|^{\alpha p s}d\mu\right)^{\frac{1}{s}}\left(\int_X |u|^{(1-\alpha)qs'}d\mu\right)^{\frac{1}{s'}}.$$

We choose s such that

$$\begin{cases} \alpha p s = p \\ (1-\alpha) q s' = q \end{cases}$$

which leads to

$$\alpha = \frac{p(q-r)}{r(q-p)}$$

or

$$\alpha = \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{p} - \frac{1}{q}}.$$

Since this last expression is equivalent to (2.6.2), we are done. □

Corollary 2.6.4. *Let $1 \leq p < d$, and $u \in W^{1,p}(\mathbb{R}^d)$. Then, for all $p \leq q \leq p^*$, we have a continuous injection $W^{1,p}(\mathbb{R}^d) \longrightarrow L^q(\mathbb{R}^d)$ and there exists a universal constant $C = C(p) < \infty$ such that*

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^d)} \leq C \|\nabla u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}. \tag{2.6.3}$$

2.6.2 Critical Case

Theorem 2.6.5. *We have a continuous embedding*

$$W^{1,d}(\mathbb{R}^d) \longrightarrow L^p(\mathbb{R}^d) \quad \text{for all } d \leq p < \infty.$$

2.6.3 Sub-Critical Case

Theorem 2.6.6. *Assume that $p > d$. Then, $W^{1,p}(\mathbb{R}^d) \hookrightarrow C^{0,\alpha} \cap L^\infty(\mathbb{R}^d)$, where $\alpha = 1 - \frac{d}{p} \in (0,1)$. Furthermore, there exists $C < \infty$ such that*

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq C \|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \tag{2.6.4}$$

and

$$|u(x) - u(y)| \leq C \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^d)} |x-y| \qquad \text{for a.e. } x, y \in \mathbb{R}^d. \tag{2.6.5}$$

2.6.4 General Result for $W^{m,p}(\Omega)$

Theorem 2.6.7. *Let $m \in \mathbb{N}$ and $1 \leq p < \infty$. We have the following results:*

1. *If $\dfrac{1}{p} - \dfrac{m}{d} > 0$, then $W^{m,p}(\mathbb{R}^d) \longrightarrow L^q(\mathbb{R}^d)$ for $q = \dfrac{d\,p}{d-p}$.*
2. *If $\dfrac{1}{p} - \dfrac{m}{d} = 0$, then $W^{m,p}(\mathbb{R}^d) \longrightarrow L^q(\mathbb{R}^d)$ for all $p \leq q < \infty$.*
3. *If $\dfrac{1}{p} - \dfrac{m}{d} < 0$, we have $W^{m,p}(\mathbb{R}^d) \longrightarrow L^\infty(\mathbb{R}^d)$. Furthermore, if $\alpha = \left(m - \frac{d}{p}\right) - \left[m - \frac{d}{p}\right] > 0$, and $k = \left[m - \frac{d}{p}\right]$, we have $u \in C^{k,\alpha}(\mathbb{R}^d)$, and for all $|\beta| \leq k$, we have*

$$|D^\beta u(x) - D^\beta u(y)| \leq C \|u\|_{\mathrm{W}^{m,p}(\mathbb{R}^d)}.$$

Proof. The proof is done by induction thanks to the previous embedding theorems, and we leave it to the reader. □

Corollary 2.6.8. *Let Ω be a bounded open subset of class C^m and assume that $\partial\Omega$ is bounded. Then, the following results hold:*

1. *If* $\dfrac{1}{p}-\dfrac{m}{d}>0$, *then* $W^{m,p}(\Omega)\hookrightarrow L^q(\Omega)$ *for* $q=\dfrac{d\,p}{d-p}$.

2. *If* $\dfrac{1}{p}-\dfrac{m}{d}=0$, *then* $W^{m,p}(\Omega)\hookrightarrow L^q(\Omega)$ *for all* $p\leq q<\infty$.

3. *If* $\dfrac{1}{p}-\dfrac{m}{d}<0$, *we have* $W^{m,p}(\Omega)\hookrightarrow L^\infty(\mathbb{R}^d)$. *Furthermore, if* $\alpha=\left(m-\frac{d}{p}\right)-\left[m-\frac{d}{p}\right]>0$, *and* $k=\left[m-\frac{d}{p}\right]$, *we have* $u\in C^{k,\alpha}(\mathbb{R}^d)$, *and for all* $|\beta|\leq k$, *and for a.e.* $x,y\in\Omega$ *such that* $B(x,2|x-y|)\cup B(y,2|x-y|)\subset\Omega$ *we have*

$$|D^\beta u(x)-D^\beta u(y)|\leq C\,\|u\|_{\mathrm{W}^{m,p}(\Omega)}\,|x-y|.$$

Remark 2.6.9. We recall that the hypothesis of C^m open subset could be weakened to Lipschitzian subset by virtue of Stein Extension Theorem.

Theorem 2.6.10 (Rellich-Kondrachov)**.** *Assume that $d\geq 2$, and that Ω is a bounded open subset of class C^1 of $\mathbb{R}^d$. Then, we have*

1. *If $p<d$, then we have a* compact embedding $W^{1,p}(\Omega)\hookrightarrow L^\infty(\Omega)$ *for all* $1\leq q<p^*$, *where* $p^*=\frac{d\,p}{d-p}$.

2. *If $p=d$, then we have a* compact embedding $W^{1,p}(\Omega)\hookrightarrow L^p(\Omega)$ *for all* $1\leq p<\infty$.

3. *If $p>d$, we have a compact embedding* $W^{1,p}(\Omega)\hookrightarrow C^0(\overline{\Omega})$.

For all $-\infty<a<b<\infty$, we have a compact embedding $W^{1,p}(]a,b[)\hookrightarrow C^0([a,b])$ for $1<p\leq\infty$ and a compact embedding $W^{1,1}(]a,b[)\hookrightarrow L^q(]a,b[)$ for all $1\leq q<\infty$.

Proof. Thanks to Ascoli's theorem, we need only treat the case $p<d$.

We apply the following compactness criterion in L^p (*[2]*, IV.25).

Theorem 2.6.11 (Riesz-Fréchet-Kolmogorov)**.** *Let Ω be an open subset of $\mathbb{R}^d$, and $U\subset\Omega$ be a relatively compact open subset. Let $\mathscr{F}$ be a bounded domain of $L^p(\Omega)$ with $1\leq p<\infty$. Assume that*

$$\forall\varepsilon>0,\ \exists\,\delta>0\ \text{such that}\ \|\tau_h f-f\|_{\mathrm{L}^p(U)}<\varepsilon\quad\forall h\in B(0,\delta)\ \text{and}\ \forall f\in\mathscr{F},$$

where $\tau_h f(x)=f(x+h)$. Then, $\mathscr{F}_{|U}$ is relatively compact in $L^p(U)$.

Fix some relatively compact open subset $U\subset\Omega$, to be determined later, and let $\varepsilon>0$. Using the interpolation inequality from Lemma (2.6.3), for all $1\leq q<p^*$, there exists $0\leq\alpha<1$ such that

$$\begin{aligned}&\|\tau_h u-u\|_{\mathrm{L}^p(U)}\leq\|\tau_h u-u\|_{\mathrm{L}^1(U)}^{\alpha}\,\|\tau_h u-u\|_{\mathrm{L}^{p^*}(U)}^{1-\alpha}\leq|h|^\alpha\,\|\nabla u\|_{\mathrm{L}^1(U)}^{\alpha}\,\|\tau_h u-u\|_{\mathrm{L}^{p^*}(U)}^{1-\alpha}\\&\leq 2^{1-\alpha}|h|^\alpha\,\|\nabla u\|_{\mathrm{L}^1(U)}^{\alpha}\,\|u\|_{\mathrm{L}^{p^*}(U)}^{1-\alpha}=C|h|^\alpha<\varepsilon\end{aligned}$$

provided that h is small enough. On the other hand, we have by Hölder's inequality

$$\|u\|_{\mathrm{L}^q(\Omega\setminus\overline{U})}\leq\|u\|_{\mathrm{L}^{p^*}(\Omega)}\left(\mathscr{L}^n(\Omega\setminus\overline{U})\right)^{1-\frac{q}{p^*}}<\varepsilon,$$

provided that $\mathscr{L}^n(\Omega\setminus\overline{U})$ is small enough.

We omit the proof of the case $d=1$ which is very similar. □

2.7 The Space $W_0^{m,p}(\Omega)$

2.7.1 Definition and first properties

Definition 2.7.1. Let $1 \leq p < \infty$. We define $W_0^{m,p}(\Omega) = \overline{\mathscr{D}(\Omega)}^{W^{m,p}}$ the closure of the space of compactly supported smooth functions in Ω for the $W^{m,p}$ topology. For $p = 2$, we write $H_0^m(\Omega) = W_0^{m,2}(\Omega)$. The space $W_0^{1,p}(\Omega)$ is a separable Banach space, and a reflexive space for $1 < p < \infty$. $H_0^m(\Omega)$ is a Hilbert space for the standard scalar product associated to $H^m(\Omega)$.

$W_0^{m,p}$ functions are functions whose traces up to the derivatives of order $m-1$ vanish on the boundary. However, in order to make the idea of trace precise, one needs to introduce fractional Sobolev spaces, that will be mentioned later in the course. We will therefore only prove two classical inequalities of fundamental importance.

Furthermore, in order to solve boundary problems for partial differential equations, the notion of trace is not formally needed in simple cases. Indeed, if $g \in W^{1,p}(\Omega)$, then we define the space

$$W_g^{1,p}(\Omega) = W^{1,p}(\Omega) \cap \left\{u : u - g \in W_0^{1,p}(\Omega)\right\}$$

of Sobolev functions whose trace on the boundary is g. This is not completely satisfactory for it requires to be able to extend g, but we will treat below the easier case of traces in $H^s(\Omega)$ (where $s \in \mathbb{R}$).

2.7.2 Poincaré Inequalities

Theorem 2.7.2 (Poincaré Inequality)**.** *Let $1 \leq p < \infty$, Ω be a bounded subset. Then, there exists a universal constant $C_P < \infty$ such that*

$$\|u\|_{\mathrm{L}^p(\Omega)} \leq C_P \|\nabla u\|_{\mathrm{L}^p(\Omega)} \qquad \text{for all } u \in W_0^{1,p}(\Omega). \tag{2.7.1}$$

Proof. Let $\varphi \in \mathscr{D}(\Omega)$, and $R > 0$ be such that $\Omega \subset \mathbb{R}^d \cap \{x : |x_d| \leq R\}$. Then, we have

$$\varphi(x', x_d) = \int_{-R}^{x_d} \partial_{x_d}\varphi(x', t)dt.$$

By Hölder's inequality, we deduce that

$$|\varphi(x', x_d)|^p \leq (2R)^{p-1} \int_{-R}^{R} |\partial_{x_d}\varphi(x', t)|^p dt.$$

Therefore, Fubini's theorem implies that

$$\int_\Omega |\varphi(x)|^p dx \leq (2R)^p \int_\Omega |\partial_{x_d}\varphi|^p dx,$$

which yields the announced inequality by density of $\mathscr{D}(\Omega)$ in $W_0^{1,p}(\Omega)$. □

Remark 2.7.3. The proof shows that the statement is true for a set that is bounded in a single direction.

Theorem 2.7.4 (Poincaré-Wirtinger Inequality)**.** *Let $1 \leq p < \infty$, and Ω be a bounded domain of $\mathbb{R}^d$. Then, there exists a universal constant $C_{PW} < \infty$ such all $u \in W^{1,p}(\Omega)$, we have*

$$\int_\Omega |u - u_\Omega|^p \leq C_{PW} \int_\Omega |\nabla u|^p dx, \tag{2.7.2}$$

where

$$u_\Omega = \fint_\Omega u \, d\mathscr{L}^d = \frac{1}{\mathscr{L}^d(\Omega)} \int_\Omega u \, d\mathscr{L}^d$$

is the mean of u on Ω.

Proof. We argue by contradiction, and let $\{u_n\}_{n\in\mathbb{N}^*} \subset W^{1,p}(\Omega)$ such that

$$\begin{aligned}&\|u_n - u_{n\Omega}\|_{\mathrm{L}^p(\Omega)} = 1\\ &\|\nabla u_n\|_{\mathrm{L}^p(\Omega)} \leq \frac{1}{n}.\end{aligned}$$

Let $v_n = u_n - u_{n\Omega}$. Then, $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$, which implies by the Rellich-Kondrachov Theorem 2.6.10 that up to a subsequence, we have $v_n \underset{n\to\infty}{\longrightarrow} v \in L^p(\Omega)$ strongly, which implies in particular that $\|v\|_{\mathrm{L}^p(\Omega)} = 1$, and $v_\Omega = 0$. However, we also have by Fatou lemma

$$\|\nabla v\|_{\mathrm{L}^p(\Omega)} \leq \liminf_{n\to\infty} \|\nabla v_n\|_{\mathrm{L}^p(\Omega)} = 0.$$

Therefore, v is constant, but the condition $v_\Omega = 0$ implies that $v = 0$, contradiction. □

2.8 The Dual Spaces $W^{-m,p'}(\Omega)$

Definition 2.8.1. For all $1 \leq p < \infty$ and $m \in \mathbb{N}$, we denote by $W^{-m,p'}(\Omega)$ the dual space of $W_0^{m,p}(\Omega)$.

Theorem 2.8.2. *For all $F \in W^{-m,p'}(\Omega)$, there exists $f_\alpha \in L^{p'}(\Omega)$ $(\alpha \in \mathbb{N}^d)$ such that*

$$\langle F, u\rangle = \sum_{|\alpha|\leq m} \int_\Omega f_\alpha\, D^\alpha u\, d\mathscr{L}^d \quad \textit{for all } u \in W_0^{m,p}(\Omega). \tag{2.8.1}$$

Remark 2.8.3. In general, the functions f_α are not unique. Notice that our previous theorem on $\mathscr{D}_{\mathrm{L}^p}(\mathbb{R}^d)$ is proven.

2.9 The Hilbert Spaces $H^s(\mathbb{R}^d)$

2.9.1 Basic Properties

Those spaces will be the first examples of interpolation spaces, and they are easy to define.

Definition 2.9.1. For all $s \in \mathbb{R}$ define

$$H^s(\mathbb{R}^d) = \mathscr{S}'(\mathbb{R}^d) \cap \left\{u : (1+|\xi|^2)^{\frac{s}{2}} \mathscr{F}(u) \in L^2(\mathbb{R}^d)\right\},$$

and equip it with the following norm:

$$\|u\|_{\mathrm{H}^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}}. \tag{2.9.1}$$

Remark 2.9.2. In the case of S^1, the space $H^s(S^1)$ is defined as follows:

$$H^s(S^1) = \mathscr{D}'(S^1) \cap \left\{u : (1+|n|)^{\frac{s}{2}} \widehat{u} \in l^2(\mathbb{Z})\right\},$$

where for all $n \in \mathbb{Z}$, we have

$$\widehat{u}(n) = \left\langle u, e^{-in\theta}\right\rangle.$$

We equip $H^s(S^1)$ with the following norm:

$$\|u\|_{\mathrm{H}^s(\mathbb{Z})} = \left(\sum_{n\in\mathbb{Z}} (1+|n|^2)^s |\widehat{u}(n)|^2\right)^{\frac{1}{2}}. \tag{2.9.2}$$

Theorem 2.9.3. *For all $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ is a separable Hilbert space, and for $m \in \mathbb{Z}$, $H^m(\mathbb{R}^d) = W^{m,2}(\mathbb{R}^d)$ with equivalent norms.*

Proof. The following quantity

$$\langle u, v \rangle_s = \int_{\mathbb{R}^d} (1+|\xi|^2)^s \widehat{u}(\xi)\overline{\widehat{v}(\xi)} d\xi \tag{2.9.3}$$

is a scalar product on H^s, and the map $u \mapsto (1+|\xi|^2)^{\frac{s}{2}}\widehat{u}$ is an isometric bijection between H^s and L^2. Since $L^2(\mathbb{R}^d)$ is complete, we deduce that H^s is complete for the norm above. We need only treat the second part in the case $m \geq 0$. By the properties of the Fourier transform, for all $u \in \mathscr{S}'(\mathbb{R}^d)$, we have $\mathscr{F}(D^\alpha u) = i^{|\alpha|}\xi^\alpha \widehat{u}$, which shows by Parseval identity that

$$\|D^\alpha u\|_{\mathrm{L}^2(\mathbb{R}^d)} = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\int_{\mathbb{R}^d} |\xi^\alpha|^2 |\widehat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}. \tag{2.9.4}$$

Notice that here exists constants $0 < C_m < \infty$ such that

$$C_m^{-1}(1+|\xi|^2)^m \leq \sum_{|\alpha| \leq m} |\xi^\alpha|^2 \leq C_m(1+|\xi|^2)^2. \tag{2.9.5}$$

Indeed, for all $|\alpha| \leq m$, we have

$$|\xi^\alpha|^2 \leq |\xi|^{2|\alpha|} \leq (1+|\xi|^2)^m,$$

while

$$\sum_{|\alpha| \leq m} |\xi^\alpha|^2 \geq 1 + \sum_{j=1}^m |\xi_j^m|^2 \geq C_1(1+|\xi|^{2m}) \geq C_2(1+|\xi|^2)^m$$

thanks to the binomial formula. Finally, we deduce by (2.9.4) and (2.9.5) that both $H^m(\mathbb{R}^d)$ norms are equivalent. ☐

Theorem 2.9.4. *$\mathscr{D}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$.*

2.9.2 Duality

Theorem 2.9.5 (Duality)**.** *For all $s \in \mathbb{R}$, for all $L \in (H^s(\mathbb{R}^d))'$, there exists a unique $v \in H^{-s}(\mathbb{R}^d)$ such that*

$$L(u) = \langle u, v \rangle = \int_{\mathbb{R}^d} u(x)\, v(x)\, dx \quad \textit{for all } u \in H^s(\mathbb{R}^d).$$

Remark 2.9.6. First, for all $(u, v) \in H^s(\mathbb{R}^d) \times H^{-s}(\mathbb{R}^d)$, we have by Parseval identity:

$$\langle u, v \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^d} \widehat{u}(\xi)\widehat{v}(-\xi) d\xi.$$

In particular, we deduce that

$$|\langle u, v \rangle| = \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^d} (1+|\xi|^2)^{\frac{s}{2}} \widehat{u}(\xi)\, (1+|\xi|^2)^{-\frac{s}{2}} \widehat{v}(\xi)\, d\xi \right| \leq \frac{1}{(2\pi)^n} \|u\|_{\mathrm{H}^s(\mathbb{R}^s)} \|v\|_{\mathrm{H}^{-s}(\mathbb{R}^d)},$$

which shows that the map $L_v : u \mapsto \langle u, v \rangle$ is a continuous linear form on $H^s(\mathbb{R}^d)$, *i.e.* an element of $(H^s(\mathbb{R}^d))'$, and that furthermore, we have

$$\|L_v\|_{(H^s(\mathbb{R}^d))'} \leq \frac{1}{(2\pi)^n} \|v\|_{\mathrm{H}^{-s}(\mathbb{R}^d)}. \tag{2.9.6}$$

The proof of the Theorem builds on this first step and Hahn-Banach theorem, and we omit it. .

2.9.3 Traces

Theorem 2.9.7. *For all $s > \frac{1}{2}$, the operator $\gamma : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^{d-1})$, such that*

$$\gamma(\varphi)(x') = \varphi(x', 0),$$

admits a unique continuous linear extension $H^s(\mathbb{R}^d) \to H^{s-\frac{1}{2}}(\mathbb{R}^d)$.

Proof. Fix some $\varphi \in \mathscr{S}(\mathbb{R}^d)$. Thanks to the Fourier inversion formula and Fubini's theorem, we have for all $x' \in \mathbb{R}^{d-1}$

$$\begin{aligned}\psi(x') = \varphi(x', 0) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{d-1} \times \mathbb{R}} \widehat{\varphi}(\xi', \xi_d) e^{i\, x' \cdot \xi'} d\xi' d\xi_d \\ &= \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} \widehat{\varphi}(\xi', \xi_d) d\xi_d \right) e^{i\, x' \cdot \xi'} d\xi'.\end{aligned}$$

Using once more the inverse Fourier formula, we deduce that

$$\widehat{\psi}(\xi') = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\varphi}(\xi', t) dt = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\widehat{\varphi}(\xi', t)(1 + |\xi'|^2 + t^2)^{\frac{s}{2}} \right) (1 + |\xi'|^2 + t^2)^{-\frac{s}{2}} dt.$$

Since $s > \frac{1}{2}$, we have

$$\int_{\mathbb{R}} \frac{dt}{(1 + |\xi'|^2 + t^2)^s} = \frac{1}{(1 + |\xi'|^2)^{s - \frac{1}{2}}} \int_{\mathbb{R}} \frac{dt}{(1 + t^2)^s} = \frac{c_s}{(1 + |\xi'|^2)^{s - \frac{1}{2}}} < \infty,$$

which implies by Cauchy-Schwarz inequality that

$$|\widehat{\psi}(\xi')|^2 \leq \frac{c_s}{(2\pi)^2} \frac{1}{(1 + |\xi'|^2)^{s - \frac{1}{2}}} \int_{\mathbb{R}} (1 + |\xi'|^2 + t^2)^s |\widehat{\varphi}(\xi', t)|^2 dt.$$

Another application of Fubini's theorem shows that

$$\begin{aligned}\|\gamma(\varphi)\|^2_{\mathrm{H}^{s-\frac{1}{2}}(\mathbb{R}^{d-1})} &= \int_{\mathbb{R}^{d-1}} (1 + |\xi'|^2)^{s - \frac{1}{2}} |\widehat{\psi}(\xi')|^2 d\xi' \\ &\leq \frac{c_s}{(2\pi)^2} \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\widehat{\varphi}(\xi)|^2 d\xi = \frac{c_s}{(2\pi)^2} \|u\|^2_{\mathrm{H}^s(\mathbb{R}^d)}\end{aligned}$$

By density of $\mathscr{S}(\mathbb{R}^d)$ in $H^s(\mathbb{R}^d)$, we deduce that $\gamma : H^s(\mathbb{R}^d) \to H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$ is a continuous linear map such that

$$\|\gamma\| \leq \frac{1}{2\pi} \left(\int_{\mathbb{R}} \frac{1}{(1 + t^2)^s} \right)^{\frac{1}{2}} = \frac{1}{\pi} \sqrt{\frac{s}{2s - 1}},$$

which concludes the proof of the theorem. □

Remark 2.9.8. In particular, if we have a continuous trace operator $H^1(B_{\mathbb{R}^2}(0,1)) \to H^{\frac{1}{2}}(S^1)$ (where $H^s(S^1)$ is defined in (2.9.2)), and more generally, the trace theorem is true for a C^1 domain, but requires to define the fractional Sobolev space, but we will only consider it on $\mathbb{R}^d$ or S^1.

Those results have applications to the solvability of the Dirichlet problem for $H^{\frac{1}{2}}$ data, which is crucial in many applications (see [1], and the exercises).

Chapter 3

Topology and Functional Spaces

3.1 Basic definitions

We assume the reader familiar with the basic notions of topology, and only recall a few basic definitions.

Definition 3.1.1. Let X be an arbitrary set. We say that $\mathscr{T} \subset \mathscr{P}(X)$ is a topology if the following properties are verified:

1. If $\{U_i\}_{i\in I} \subset \mathscr{T}$ is an arbitrary family of elements of $\mathscr{T}$, then $\displaystyle\bigcup_{i\in I} U_i \in \mathscr{T}$ **(stability by arbitrary union)**
2. If $U_1, \cdots, U_n \in \mathscr{T}$, then $\displaystyle\bigcap_{i=1}^{n} U_i \in \mathscr{T}$ **(stability by finite intersection).**

Elements of $\mathscr{T}$ are called *open sets*, and complements of open sets are called *closed sets.* We say that such a couple $(X, \mathscr{T})$ is a *topological space.*

Remark 3.1.2. Notice that a set may be closed and open. Taking an empty union and empty intersection, we deduce that both $\varnothing$ and X are open sets, which implies by definition that they are closed too.

On a non-empty set X, there are always at least two topologies: the *trivial topology* given by $\mathscr{T} = \{\varnothing, X\}$, and the *discrete topology* given by $\mathscr{T} = \mathscr{P}(X)$.

We will need of the notion of basis of topology later.

Definition-Proposition 3.1.3. *Let $\mathscr{T}_0 = \{U_i\}_{i\in I}$ be a non-empty collection of sets of a non-empty set X. The smallest topology $\mathscr{T}$ that contains $\mathscr{T}_0$ is given by the following construction. Let $\mathscr{T}_1$ be the family of finite intersection of $\mathscr{T}_0$. Then, $\mathscr{T}$ is given by*

$$\mathscr{T} = \mathscr{P}(X) \cap \left\{ W : W = \bigcup_{j\in J} V_j, V_j \in \mathscr{T}_1 \text{ for all } j \in J \right\}. \tag{3.1.1}$$

Proof. Notice that an arbitrary intersection of topologies is a topology. Indeed, let $\{\mathscr{T}_j\}_{j\in J}$ be a family of topologies, and consider $\mathscr{T} = \displaystyle\bigcap_{j\in J} \mathscr{T}_j$, and $\{U_i\}_{i\in I} \subset \mathscr{T}$. In particular, we have $\displaystyle\bigcup_{i\in I} U_i \in \mathscr{T}_j$ for all $j \in J$, which implies that $\displaystyle\bigcup_{i\in I} U_i \in \mathscr{T}$. Therefore, $\mathscr{T}$ is well-defined and given by

$$\mathscr{T}' = \bigcap_{\mathscr{T}'' \text{ topology } \mathscr{T}_0 \subset \mathscr{T}''} \mathscr{T}'',$$

which is a topology by the above proof. Now, we need to show that $\mathscr{T} = \mathscr{T}'$. Notice that we trivially have $\mathscr{T} \subset \mathscr{T}'$ by using both defining properties of topologies. Therefore, we need only show that $\mathscr{T}$ is a topology to conclude the proof. By construction, $\mathscr{T}$ is stable by arbitrary unions, so we only have to check that $\mathscr{T}$ is stable under finite intersection. Let $W_1, \cdots, W_n \in \mathscr{T}$. Then, there exists sets $J_1, \cdots, J_n$ and $V_{i,j_i} \in \mathscr{T}_1$ ($j_i \in J_i$, $1 \leq i \leq n$) such that

$$W_i = \bigcup_{j_i \in J_i} V_{i,j_i}.$$

Furthermore, we have $V_{i,j_i} = U_{i,j_i}^1 \cap \cdots \cap U_{i,j_i}^{k_i}$ for some $U_{i,j_i}^k \in \mathscr{T}$. Finally, we deduce that

$$W_1 \cap \cdots \cap W_n = \bigcap_{i=1}^n \bigcup_{j_i \in J_i} \bigcap_{k=1}^{k_i} U_{i,j_i}^k.$$

Let $x \in W_1 \cap \cdots \cap W_n$. Then, for all $1 \leq i \leq n$, there exists $j_i \in J_i$ such that $x \in U_{i,j_i}^1 \cap \cdots \cap U_{i,j_i}^{k_i}$. In particular, we have

$$x \in \bigcap_{i=1}^n \left(\bigcap_{k=1}^{k_i} U_{i,j_i}^k\right),$$

and

$$W_1 \cap \cdots \cap W_n \in W = \bigcup_{(j_1, \cdots, j_n) \in J_1 \times \cdots \times J_n} \bigcap_{i=1}^n \left(\bigcap_{k=1}^{k_i} U_{i,j_i}^k\right) \in \mathscr{T}_1.$$

Likewise, if $x \in W$, then there exists $(j_1, \cdots, j_n) \in J_1 \times \cdots \times J_n$ such that

$$x \in \bigcap_{i=1}^n \left(\bigcap_{k=1}^{k_i} U_{i,j_i}^k\right).$$

A fortiori, we have

$$x \in \bigcap_{i=1}^n \bigcup_{j_i \in J_i} \bigcap_{k=1}^{k_i} U_{i,j_i}^k = W_1 \cap \cdots \cap W_n,$$

which proves that $W = W_1 \cap \cdots \cap W_n \in \mathscr{T}_1$ and that $\mathscr{T}$ is a topology on X. □

Let us also recall the fundamental notion of neighbourhood.

Definition 3.1.4. Let $(X, \mathscr{T})$ be a topological space. We say that a (non-empty) set N is a neighbourhood of a point $x \in X$ if there exists an open set U containing x such that $U \subset N$.

Finally, we also need the basic notion of interior, closure and frontier of a set.

Definition 3.1.5. Let $(X, \mathscr{T})$ be a topological space. Let $A \subset X$. Its *interior*, denoted by $\mathrm{int}(A)$ or $\mathring{A}$, is the largest open set contained in A, given explicitly by

$$\mathrm{int}(A) = \bigcup_{U \subset A, U \in \mathscr{T}} U,$$

whilst the *closure* of A, denoted by $\mathrm{clos}(A)$ or $\overline{A}$, is the smallest closed set containing A, given explicitly by

$$\mathrm{clos}(A) = \bigcap_{F \supset A, X \setminus F \in \mathscr{T}} F.$$

The frontier (or boundary) of A is given by $\partial A = \overline{A} \setminus \mathrm{int}(A)$.

The defining properties of a topology trivially imply that those notions are well-defined for the arbitrary intersection of closed sets is closed. Those definitions show that arbitrary unions are in general needed to perform basic operations that mimic the classical notions in Euclidean spaces and manifolds.

The following notion will prove crucial in many a proof of those lectures. Indeed, proofs are typically much easier for smooth or more regular functions, and when those functions are dense in a given (Banach) space of functions, a standard argument typically allows one to extend the proof from smooth functions to arbitrary functions in the said Banach space.

Definition 3.1.6. We say that a subset $A \subset X$ of a topological space $(X, \mathscr{T})$ is dense if $\overline{A} = X$.

We say that X is separable if it admits a countable dense set.

Finally, recall the notion of continuity.

Definition 3.1.7. Let $(X, \mathscr{T})$, $(Y, \mathscr{S})$ be two topological spaces. We say that a map $f : X \to Y$ is continuous if for all open set $V \in \mathscr{S}$, we have $f^{-1}(V) \in \mathscr{T}$.

We can finally move to the familiar concept of metric spaces (all spaces encountered in this lecture are metrisable).

Definition 3.1.8. Let X be an arbitrary set. We say that a map $d : X \times X \to \mathbb{R}_+$ is a metric if the following three properties are satisfied

1. $d(x, y) = 0$ if and only if $y = x$ **(definiteness)**.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$ **(symmetry)**.
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ **(triangle inequality)**.

If d is a metric on X, the *open ball* of centre $x \in X$ and radius $r > 0$ is defined by $B(x, r) = X \cap \{y : d(x, y) < r\}$, and the closed ball by $\overline{B}(x, r) = X \cap \{y : d(x, y) \leq r\}$.

Definition 3.1.9. A metric space (X, d) is a topological space whose basis of open sets is given by the sets of all open balls $\{B(x, r)\}_{x \in X, r > 0}$.

Remark 3.1.10. Notice that metric spaces are always separated. It is quite unfortunate choice of terminology, for the closed ball in an arbitrary metric is not always closed. However, the closed ball is always closed in a normed space.

Theorem 3.1.11. *Let (X, d) and (Y, h) be two metric spaces. Then $f : X \to Y$ is continuous if and only if f is sequentially continuous,* i.e. *for all $x \in X$ and for all sequences $\{x_n\}_{n \in \mathbb{N}}$ such that $x_n \underset{n \to \infty}{\longrightarrow} x$, we have $f(x_n) \underset{n \to \infty}{\longrightarrow} f(x) \in Y$.*

We can now move on to the definition of normed space, Banach space, and Hilbert space.

Definition 3.1.12. **1**. Let X be a vector space on a field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). We say that a map $\|\cdot\|_X : X \to \mathbb{R}$ is a norm if the following associated map $d : X \to X \to \mathbb{R}_+$, such that $d_X(x, y) = \|x - y\|_X$ is a distance on X, and for all $\lambda \in \mathbb{K}$, we have

$$\|\lambda\, x\|_X = |\lambda|\, \|x\|_X\,.$$

The metric space (X, d_X) is called a *normed space* and denoted (abusively) $(X, \|\cdot\|_X)$.

2. We say that $(X, \|\cdot\|_X)$ is a Banach space if the metric space $(X, \|\cdot\|_X)$ is a complete metric space.

In the following, $\mathbb{K}$ will denote either $\mathbb{R}$ or $\mathbb{C}$.

Remarks 3.1.13. **1**. Notice that we have by the triangle inequality for all $x, y \in X$

$$\|x + y\| = d(x, -y) \leq d(x, 0) + d(0, -y) = \|x\| + \|y\|\,.$$

2. In reality, there are no abuses of notations for the distance associated to a norm is defined bi-univocally.

We can now move to the definition of Hilbert spaces. We first need to remind the definition of scalar product.

Definition 3.1.14. Let E be a vector space on $\mathbb{K}$. A scalar product $\langle\,\cdot\,,\,\cdot\,\rangle : E\times E\rightarrow\mathbb{R}$ is a positive-definite symmetric bilinear functional. In other words, it satisfies the following properties:

1. $\langle x,x\rangle>0$ for all $x\in E\setminus\{0\}$ **(positive-definiteness)**.
2. $\langle x,y\rangle=\overline{\langle y,x\rangle}$ for all $x,y\in E$ **(conjugate symmetry)**.
3. $\langle\lambda\,u+v,w\rangle=\lambda\langle u,w\rangle+\langle v,w\rangle$ for all $u,v,w\in E$ and $\lambda\in\mathbb{K}$ **(linearity in the first variable)**.

Remark 3.1.15. Since $\langle\,\cdot\,,\,\cdot\,\rangle$ is symmetric, we need only check the linearity in the first variable.

Definition 3.1.16. We say that a Banach space $(H,\|\cdot\|)$ is a Hilbert space if the following functional

$$\langle x,y\rangle=\frac{1}{4}\left(\|x+y\|^2-\|x-y\|^2\right),\qquad x,y\in H$$

is a scalar product on H.

We do not recall here the useful properties of Hilbert spaces (Riesz-Fréchet representation theorem, Hilbertian basis, and spectral decomposition that will not play a role immediately).

Remark 3.1.17. It may seem that we are replacing a definition by a theorem, but the polarisation formula shows that it is a trivially equivalent definition.

Before mentioning the notion of dual space of a normed space and weak topology, let us recall a statement of the Hahn-Banach theorem (see [2]).

Theorem 3.1.18 (Hahn-Banach)**.** *Let X be a real vector space and $N:X\rightarrow\mathbb{R}$ be a sub-linear homogenous map of degree* 1*,* i.e. *a map such that*

1. $N(\lambda\,x)=\lambda\,N(x)$ *for all $x\in X$ and $\lambda>0$.*
2. $N(x+y)\leq N(x)+N(y)$ *for all $x,y\in X$.*

*Let $Y\subset X$ be a sub-vector space, and $f:Y\rightarrow\mathbb{R}$ be a linear map such that $f\leq N_{|Y}$. Then, there exists an extension $\overline{f}:X\rightarrow\mathbb{R}$—*i.e. *such that $\overline{f}_{|Y}=f$—such that $\overline{f}\leq N$ on X.*

The proof uses the axiom of choice, and more precisely, the equivalent formulation known as the Zorn's lemma.* First introduce the following definitions.

Definition 3.1.19. (i) A partial order on a set X is a binary relation $\leq$ on $X\times X$ that satisfies the following properties:

1. $x\leq x$ for all $x\in X$ **(reflexivity)**.
2. For all $x,y\in X$, if $x\leq y$ and $y\leq x$, then $x=y$ **(anti-symmetry)**.
3. For all x,y,z, if $x\leq y$ and $y\leq z$, then $x\leq z$ **(transitivity)**.

(ii) We say that a subset $Y\subset X$ is totally ordered (by $\leq$) if for all $x,y\in Y$, we have either $x\leq y$, or $y\leq x$—in which case, we say that $\leq$ is a *total order* (on Y).
(iii) We say that an element $x\in X$ is an upper bound of Y is $y\leq x$ for all $y\in X$.
(iv) Finally, we say that $x\in X$ is a maximal element if for all $y\in X$ such that $x\leq y$, we have $y=x$.

*Another equivalent statement for the axiom of choice is *Zermelo's Theorem*, that asserts that any set can be *well-ordered.* This terminology is rather poorly chosen for what is called either a lemma or a theorem is nothing else than an axiom. However, more than a century of usage will not be erased easily.

Lemma 3.1.20 (Zorn's lemma)**.** *Let* $(X, \leq)$ *be a non-empty inductive set,* i.e. *a set such that every totally ordered subset admits an upper bound. Then,* X *admits a maximal element.*

We can finally prove the Hahn-Banach theorem.

Proof. (Of Theorem 3.1.18)

Step 1. Finite-dimensional case.

The theorem is true in finite dimension without the axiom of choice, so let us first prove that a linear map $f : \mathbb{R}^k \subset \mathbb{R}^n \to \mathbb{R}$ (where $k < n$) always admits an extension $\overline{f}$ to $\mathbb{R}^{k+1}$ satisfying $\overline{f} \leq N$ on $\mathbb{R}^{k+1}$. Seeing $\mathbb{R}^k$ as $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$, we extend f by $\overline{f} : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$ by

$$\overline{f}(x,t) = f(x) + \alpha\, t \qquad \text{for all } (x,t) \in \mathbb{R}^k \times \mathbb{R},$$

for some $\alpha \in \mathbb{R}$ to be determined later. For all $(x,t) \in \mathbb{R}^{k+1}$, we must have

$$f(x) + \alpha\, t \leq N(x,t),$$

where we identify by abuse of notation (x,t) with $(x,,t,0) \in \mathbb{R}^n$. For $t > 0$, by homogeneity of N, the inequality is equivalent to

$$\left(f(x) + \alpha\, t \leq tN(t^{-1}x, 1)\right) \Longleftrightarrow \left(f(y) + \alpha \leq N(y,1) \quad (y = t^{-1}x)\right),$$

and for $t < 0$, we get the condition

$$f(y) - \alpha \leq N(y, -1).$$

Therefore, α must satisfy

$$\sup_{y \in \mathbb{R}^k} \left(f(y) - N(y,-1)\right) \leq \alpha \leq \inf_{z \in \mathbb{R}^k} (-f(z) + N(z,1)).$$

Such an α always exists for $f(y) - N(y,-1) \leq -f(z) + N(z,1)$ for all $y, z \in \mathbb{R}^k$. Indeed, we have by linearity of f

$$f(y) + f(z) = f(y+z) \leq N(y+z) = N(y+z, -1+1) \leq N(y,-1) + N(z,1),$$

which concludes the proof of this step. Notice that an immediate induction gives an extension of f to $\mathbb{R}^n$.

Step 2. General case.

Let E be the set of extensions $g : D(g) \to \mathbb{R}$ of f (where $D(g) \supset Y$ is the domain of g) such that $g \leq N|_{D(g)}$. We introduce the partial order relation $\leq$ on E as follows:

$$(g_1 \leq g_2) \Longleftrightarrow \left(D(g_1) \subset D(g_2) \text{ and } g_2 = g_1 \text{ on } D(g_1)\right).$$

The set E is not empty since $f \in E$. Furthermore if $F \subset E$ is totally ordered, writing $F = \{g_i\}_{i \in I}$, we see that $g : \bigcup_{i \in I} D(g_i) \to \mathbb{R}$ such that $g = g_i$ on $D(g_i)$ is a well-defined function and an upper bound of F. Therefore, E is inductive, and admits a maximal element that we will denote by f_0. By **Step 1**, if $D(f_0) \neq X$, f_0 admits an extension $\overline{f}_0 : D(\overline{f}_0) \to \mathbb{R}$ such that $D(\overline{f}_0)/D(f_0) \simeq \mathbb{R}$ has codimension 1. In particular, it would imply that f_0 is not a maximal element, a contradiction. Therefore, $D(f_0) = X$ and $\overline{f} = f_0$ is an extension of f satisfying all expected properties. □

Remark 3.1.21. Notice that we do not use the finite-dimension of the ambient space $\mathbb{R}^n$ in **Step 1**, and this why we can apply it to the (potentially) infinite-dimensional case of **Step 2**.

We now let in the rest of this chapter $(X, \|\cdot\|)$ be a normed space. The dual space X' (or X^*) is the space of continuous linear forms $f : X \to \mathbb{R}$ equipped with the following dual norm

$$\|f\|_{X'} = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |f(x)|. \tag{3.1.2}$$

From Hahn-Banach theorem, we deduce the following corollary.

Corollary 3.1.22. *Let $Y \subset X$ be a sub-vector space, and $f : Y \to \mathbb{R}$ be a continuous linear form. Then, there exists an extension $\overline{f} : X \to \mathbb{R}$ such that $\|\overline{f}\|_{X'} = \|f\|_{Y'}$.*

Proof. Take $N(x) = \|f\|_{Y'}\,\|x\|$. ☐

Corollary 3.1.23. *For all $x \in X$, there exists $f \in X'$ such that $\|f\|_{X'} = \|x\|_X$ and $f(x) = \|x\|_X^2$.*

Proof. Apply Corollary 3.1.22 to $f_0 : \mathbb{R}\,x \to \mathbb{R}, t \mapsto \|x\|_X^2\, t$. ☐

Corollary 3.1.24. *For all $x \in X$, we have*

$$\|x\|_X = \sup_{\substack{f\in X'\\ \|f\|_{X'}\leq 1}} |f(x)| = \max_{\substack{f\in X'\\ \|f\|_{X'}\leq 1}} |f(x)|. \tag{3.1.3}$$

Proof. The inequality $|f(x)| \leq \|f\|_{X'}\,\|x\|_X$ and Corollary 3.1.23 imply the result immediately. ☐

We will not mention other the geometric forms of Hahn-Banach theorem (see [2]), but we will need the following very useful result in the rest of the lecture.

Theorem 3.1.25. *Let $Y \subset X$ be a sub-vector space such that $\overline{Y} \neq X$. Then, there exists $f \in X' \setminus \{0\}$ such that $f_{|Y} = 0$.*

3.2 The Three Fundamental Theorem of Linear Operators in Banach Spaces

First recall the Baire lemma.

Lemma 3.2.1 (Baire)**.** *Let (X, d) be a complete metric space. Let $\{F_n\}_{n\in\mathbb{N}} \subset X$ a sequence of closed spaces of empty interior,* i.e. *such that* $\mathrm{int}(F_n) = \varnothing$ *for all $n \in \mathbb{N}$. Then, $\bigcup_{n\in\mathbb{N}} F_n$ has empty interior too.*

Let Y be a normed vector space. We denote by $\mathscr{L}(X, Y)$ the space of continuous linear operators $X \to Y$, equipped with the following norm

$$\|T\|_{\mathscr{L}(X,Y)} = \sup_{\substack{x\in X\\ \|x\|_X\leq 1}} \|T(x)\|_Y .$$

We skip the standard proof by induction, and we simply recall the main theorems of Banach spaces.

Theorem 3.2.2 (Banach-Steinhaus, or Principle of Uniform Boundedness)**.** *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, be two Banach spaces, and $\{T_i\}_{i\in I} \subset \mathscr{L}(X, Y)$ be a family of continuous linear operators from X into Y. Assume that for all $x \in X$, we have*

$$\sup_{i\in I} \|T_i(x)\|_Y < \infty. \tag{3.2.1}$$

Then, we have

$$\sup_{i\in I} \|T_i\|_{\mathscr{L}(X,Y)} < \infty. \tag{3.2.2}$$

Proof. For all $n \in \mathbb{N}$, let $F_n = X \cap \{x : \forall i \in I, \|T_i(x)\| \leq n\}$. Then F_n is an intersection of closed sets, therefore, a closed set. Furthermore, we have $\bigcup_{n\in\mathbb{N}} F_n = X$. Therefore, by Baire's lemma, we deduce that there exists $N \in \mathbb{N}$ such that $\mathrm{int}(F_N) \neq \varnothing$. In particular, there exists an open ball $B(x_0, r)$ in F_N, and we deduce that

$$\forall i \in I,\ \|T_i(x - x_0)\|_Y \leq N \qquad \text{for all } x \in B(x_0, r).$$

By linearity, we deduce that

$$\forall i \in I,\ \|T_i(x)\|_Y \leq \frac{1}{r}\left(N + \|T_i(x_0)\|_Y\right)\|x\|_X \leq C\,\|x\|_X\,,$$

using (3.2.1) with $x = x_0$. □

Let us list a few corollaries.

Corollary 3.2.3. *Let X and Y be two Banach spaces. Let $\{T_n\}_{n\in\mathbb{N}} \subset \mathscr{L}(X,Y)$ be a sequence of linear continuous operators from X to Y, such that for all $x \in X$, the sequence $\{T_n(x)\}_{n\in\mathbb{N}} \subset Y$ converges to a limit denoted by $T(x) \in Y$. Then, the following properties are satisfied:*

1. $\sup_{n\in\mathbb{N}} \|T_n\|_{\mathscr{L}(X,Y)} < \infty$.

2. $T \in \mathscr{L}(X,Y)$.

3. $\|T\|_{\mathscr{L}(X,Y)} \leq \liminf_{n\to\infty} \|T_n\|_{\mathscr{L}(X,Y)}$.

Proof. The first point **1**. follows from Theorem 3.2.2. In particular, there exists a constant $C < \infty$ such that

$$\sup_{n\in\mathbb{N}} \|T_n(x)\|_Y \leq C\,\|x\|_X \quad \text{for all } x \in X.$$

In particular, we have

$$\|T(x)\|_Y \leq C\,\|x\|_X \quad \text{for all } x \in X.$$

By linearity of T_n, we deduce that T is linear, which proves **2**. Finally, the inequality

$$\|T_n(x)\| \leq \|T_n\|_{\mathscr{L}(X,Y)}\,\|x\|_X \quad \text{for all } x \in X$$

implies the last point **3**. □

Corollary 3.2.4. *Let X be a Banach space and $A \subset X$ an arbitrary subset. Assume that A is weakly bounded,* i.e. *for all $f \in X'$, the set $f(A) \subset \mathbb{R}$ is bounded. Then, A is strongly bounded in X.*

Proof. Let $\{T_a\}_{a\in A} \subset \mathscr{L}(X',\mathbb{R})$ be defined by $T_a(f) = f(a)$ for all $f \in X'$. Then, we have

$$\sup_{a\in A} \|T_a(f)\| < \infty \quad \text{for all } f \in X'.$$

Therefore, by Theorem 3.2.2, we have

$$\sup_{a\in A} \|T_a\|_{\mathscr{L}(X',\mathbb{R})} < \infty.$$

In particular, we have

$$|f(a)| \leq C\,\|f\|_{X'} \ \text{ for all } f \in X'.$$

Using Corollary 3.1.23, we deduce that $\|a\| \leq C$ for all $a \in A$, which concludes the proof. □

The dual statement is given by the following.

Corollary 3.2.5. *Let X be a Banach space and $F \subset X'$. Assume that for all $x \in X$, the set $F(x) = \mathbb{R} \cap \{y : y = f(x)$ for some $f \in F\}$ is bounded. Then, F is bounded.*

Proof. The proof is almost identical, using the family $\{T_f = f\}_{f\in F}$. □

The second fundamental theorem of Banach is the following.

Theorem 3.2.6 (Open Mapping Theorem)**.** *Let X and Y be two Banach spaces, and $T \in \mathscr{L}(X,Y)$ be a surjective linear continuous operator. Then, there exists $r > 0$ such that*

$$B_Y(0,r) \subset T(B_X(0,1)).$$

We skip the long proof.

Finally, we give the third theorem of Banach.

Theorem 3.2.7 (Closed Graph Theorem)**.** *Let X, Y be two Banach spaces, and $T : X \to Y$ be a linear map. Assume that the graph of T,*

$$\mathscr{G}(T) = X \times Y \cap \{(x,y) : y = T(x)\}$$

is a closed set of $X \times Y$. Then T is continuous.

Proof. Consider on X the norm $|||x||| = \|x\|_X + \|T(x)\|_Y$. Since $\mathscr{G}(T)$ is closed, $(X, |||\cdot|||)$ is a Banach space. Furthermore, we have $\|\cdot\|_X \leq |||\cdot|||$. By the open map theorem applied to the identity map $(X, |||\cdot|||) \to (X, \|\cdot\|)$, we deduce that there exists $r > 0$ such that

$$r|||x||| \leq \|x\| \quad \text{for all } x \in B_{(X,|||\cdot|||)}(0,1),$$

which shows that $\|T\|_{\mathscr{L}(X,Y)} \leq \frac{1}{r} - 1 < \infty$. □

The argument in the second part of the proof works in a more general setting.

Corollary 3.2.8. *Let X and Y be two Banach spaces, and let $T \in \mathscr{L}(X,Y)$ be a bijective linear operator. Then, the inverse $T^{-1} : Y \to X$ is continuous.*

Proof. By the Open Mapping Theorem (Theorem 3.2.6), we deduce that there exists $r > 0$ such that

$$r\,\|x\|_X \leq \|T(x)\|_Y \quad \text{for all } x \in B_X(0,1),$$

which shows that $\left\|T^{-1}\right\|_{\mathscr{L}(Y,X)} \leq \frac{1}{r}$. □

3.3 Weak Topology

3.3.1 General Definition

Let X be a set and $\{Y_i\}_{i\in I}$ be a family of topological spaces. For all $i \in I$, we fix some map $\varphi_i : X \to Y$. The *weak topology* on X is with topology that makes all maps $\varphi_i : X \to Y_i$ continuous. Notice that this is well-defined by Definition 3.1.3), and the associated pre-topology is given by $\mathscr{T}_0 = \left\{\varphi_i^{-1}(V_i) : V_i \text{ open subset of } Y_i\right\}$.

Proposition 3.3.1. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of X. Then $x_n \underset{n\to\infty}{\rightharpoonup} x$ for the weak topology if and only if $\varphi_i(x_n) \underset{n\to\infty}{\longrightarrow} \varphi_i(x) \in Y_i$ for all $i \in I$.*

Proof. The first implication is trivial for each map $\varphi_i : X \to Y_i$ is continuous with respect to the weak topology. Conversely, let U be a neighbourhood of x. By the construction of Definition 3.1.3, we can assume that

$$U = \bigcap_{j=1}^{n} \varphi_{i_j}^{-1}(V_{i_j}),$$

where V_{i_j} is an open set of Y_{i_j} (by hypothesis, V_{i_j} is also a neighbourhood of $\varphi_{i_j}(x)$). For all $1 \leq j \leq n$, there exists $N_j \in \mathbb{N}$ such that $\varphi_{i_j}(x_n) \in V_{i_j}$ for al $n \geq N_j$. In particular, taking $N = \max\{N_1, \cdots, N_n\}$, we deduce that for all $n \geq N$, $x_n \in U$, which concludes the proof. □

3.3.2 Weak Topology on a Banach Space

Let X be a Banach space, and $f \in X'$. Let $\varphi_f : X \to \mathbb{R}$ be defined by $\varphi_f(x) = f(x)$ for all $x \in X$. Then, the weak topology $\sigma(X, X')$ on X is the weak topology associated to the family of maps $\{\varphi_f\}_{f \in X'}$. To emphasise the duality, we will sometimes write $f(x) = \langle f, x \rangle$.

We will denote the weak convergence of $\{x_n\}_{n\in\mathbb{N}} \subset X$ to some element $x \in X$ in the weak topology by the half-arrow $\rightharpoonup$. Notice that by what precedes (Proposition 3.3.1), we have

$$\left(x_n \underset{n\to\infty}{\rightharpoonup} x\right) \Longleftrightarrow \left(f(x_n) \underset{n\to\infty}{\longrightarrow} f(x) \in \mathbb{R} \text{ for all } f \in X'\right).$$

Let us list some basic properties of the weak topology.

Proposition 3.3.2. *The weak topology $\sigma(X, X')$ is separated.*

Proof. The proof follows from the geometric version of Hahn-Banach theorem, and will be omitted. □

Proposition 3.3.3. *Let $\{x_n\}_{n\in\mathbb{N}} \subset X$. The following properties are verified:*

1. *The sequence $\{x_n\}_{n\in\mathbb{N}}$ weakly converges to some element $x \in X$ if and only if $f(x_n) \underset{n\to\infty}{\longrightarrow} f(x) \in \mathbb{R}$ for all $f \in X'$.*
2. *If $x_n \underset{n\to\infty}{\longrightarrow} x$ strongly, then $x_n \underset{n\to\infty}{\rightharpoonup} x$ weakly.*
3. *If $x_n \underset{n\to\infty}{\rightharpoonup} x$ weakly, then $\{\|x_n\|\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ is bounded and*
$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|. \tag{3.3.1}$$
4. *If $x_n \underset{n\to\infty}{\rightharpoonup} x$ weakly, and $\{f_n\}_{n\in\mathbb{N}} \subset X'$ converges towards some element $f \in X'$, then $f_n(x_n) \underset{n\to\infty}{\longrightarrow} f(x)$.*

Proof. The first property **1**. is trivial by definition of the weak topology and Proposition 3.3.1. The second **2**. follows from the inequality $|f(x_n) - f(x)| \leq \|f\|_{X'} \|x_n - x\|_X$.

Let us prove **3**. now. We apply Corollary 3.2.5. We need to check that for all $f \in X'$, $\{f(x_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded, which is trivially satisfied since $f(x_n) \underset{n\to\infty}{\longrightarrow} f(x)$ by definition of the weak convergence. Furthermore, for all $n \in \mathbb{N}$, we have

$$|f(x_n)| \leq \|f\|_{X'} \|x_n\|_X,$$

which implies that

$$|f(x)| \leq \|f\|_{X'} \liminf_{n\to\infty} \|x_n\|_X.$$

Finally, Corollary 3.1.24 implies that

$$\|x\|_X = \max_{\substack{f\in X' \\ \|f\|_{X'}\leq 1}} |f(x)| \leq \liminf_{n\to\infty} \|x_n\|_X.$$

The last property **4**. follows immediately by the triangle inequality:

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \leq \|f_n - f\|_{X'} \|x_n\|_X + \|f\|_{X'} \|x_n - x\|_X,$$

which implies the claim by the previous property **3**.. □

We end this section by a few remarks on the weak topology.

Remarks 3.3.4. The weak topology has many surprising properties:

1. The adherence of the unit sphere $S = X \cap \{x : \|x\|_X = 1\}$ for the weak topology is the closed ball $\overline{B} = X \cap \{x : \|x\|_X \leq 1\}$. We will see that in a reflexive space (to be defined in Definition 3.4.1), $\overline{B}$ is a *compact* set for the weak topology, although this set is never compact for the strong topology in infinite dimension. This is why the weak topology is so important: it allows one to solve partial differential equations thanks to a compactness argument.

2. The interior of $B = X \cap \{x : \|x\|_X < 1\}$ for the weak topology is empty.

3. In infinite dimension, the weak topology is never metrisable. This is why it is futile to define it using convergence of sequences, although for most applications, one need only look at sequences.

4. In infinite dimension, there are sequences that converge weakly but do not converge strongly.

3.4 Weak $*$ Topology

Let X be a Banach space, X' its dual space, and $X'' = (X')'$ the dual space of X' (also called *bidual* of X). We endow it with the following norm

$$\|\varphi\|_{X''} = \sup_{\substack{f \in X' \\ \|f\|_{X'} \leq 1}} |\varphi(f)|. \tag{3.4.1}$$

There is a canonical injection $J : X \to X''$, defined as follows. Let $x \in X$ and $J(x) : X' \to \mathbb{R}, f \mapsto \langle J(x), f\rangle = f(x)$. Then $J(x) \in X''$. Furthermore, we immediately check that J defines a linear map $X \to X''$, which is an isometry for

$$\|J(x)\|_{X''} = \sup_{\substack{f \in X' \\ \|f\|_{X'} \leq 1}} |\langle J(x), f\rangle| = \sup_{\substack{f \in X' \\ \|f\|_{X'} \leq 1}} |f(x)| = \|x\|_X\,, \tag{3.4.2}$$

using Corollary 3.1.23.

Therefore, X is isometric to a subset of X''. This allows us to introduce a fundamental notion that will prove fundamental in the following (and explain all the pathologies of spaces such as L^1 and L^∞).

Definition 3.4.1 (Reflexive spaces)**.** We say that a Banach space is reflexive if the isometric injection $J : X \hookrightarrow X''$ is surjective, *i.e.* $J(X) = X''$.

Common examples of reflexive spaces are the L^p spaces (on a locally compact group, say) for exponents $1 < p < \infty$.

Before listing the major properties of reflexive spaces, we now define the weak $*$ topology $\sigma(X', X)$ on X'.

Definition 3.4.2. The weak $*$ topology[*] is the smallest topology that makes all maps $J(x) : X' \to \mathbb{R}$ continuous, where $x \in X$. We denote it $\sigma(X', X)$. We denote by the convergence for sequences of X'.

Let us give a few basic properties of the weak topology.

Proposition 3.4.3. *The weak $*$ topology on X' is separated.*

Proof. Let $f, g \in X'$ such that $f \neq g$. Then, there exists $x \in X$ such that $f(x) \neq g(x)$. Assume without loss of generality that $f(x) < g(x)$, and let $\alpha \in \mathbb{R}$ such that

$$f(x) < \alpha < g(x),$$

then $J(x)^{-1}(]-\infty, \alpha[)$ and $J(x)^{-1}(]\alpha, \infty[)$ are disjoint open (for the weak $*$ topology) subset of X' that respectively contain f and g. $\square$

[*]One pronounces *weak star topology.*

We now list the basic properties of the weak $*$ topology (the proof is almost identical to the one of Proposition 3.3.3, and we omit it).

Proposition 3.4.4. *Let $\{f_n\}_{n\in\mathbb{N}} \subset X'$. Then the following properties are satisfied.*

1. *The sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to $f \in X'$ if and only if $f_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$ for all $x \in X$.*
2. *If $f_n \underset{n\to\infty}{\longrightarrow} f \in X'$ strongly, then $f_n \underset{n\to\infty}{\rightharpoonup} f$ weakly for the weak topology $\sigma(X', X'')$. If $f_n \underset{n\to\infty}{\rightharpoonup} f \in X'$ for the weak topology $\sigma(X', X'')$, then $f_n \underset{n\to\infty}{\rightharpoonup} f$ weakly for the weak $*$ topology $\sigma(X', X)$.*
3. *If $f_n n \to \infty f$, then $\{\|f_n\|_{X'}\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ is bounded and*
$$\|f\|_{X'} \leq \liminf_{n\to\infty} \|f_n\|_{X'}\,. \tag{3.4.3}$$
4. *If $f_n n \to \infty f$, and $x_n \underset{n\to\infty}{\longrightarrow} x$ strongly, then $f_n(x_n) \underset{n\to\infty}{\longrightarrow} f(x)$.*

We end this section by a fundamental compactness theorem that justifies the introduction.

Theorem 3.4.5 (Banach-Alaoglu-Bourbaki)**.** *The unit closed ball $B = X' \cap \{f : \|f\|_{X'} \leq 1\}$ is compact for the weak $*$ topology $\sigma(X', X)$.*

Proof. The proof is an easy application of Tychonoff's theorem (the arbitrary product of compact set is compact). Notice that this "theorem" is equivalent to the axiom of choice, so it was not very limiting to use Hahn-Banach theorem previously considering that the compactness of the unit ball for the weak $*$ topology is needed in many applications.

Now, let $Y = \mathbb{R}^X$ equipped with the product topology. Let $\Phi : X' \to Y$ defined by

$$\Phi(f) = \{f(x)\}_{x\in X} \quad \text{for all } \ f \in X'.$$

By definition of the product topology, since each canonical projection $\pi_x \circ \Phi = J(x) : X' \to \mathbb{R}$ is continuous ($x \in X$), we deduce that Φ is a continuous map. Here, we denoted $\pi_x : Y = \mathbb{R}^X \to \mathbb{R}$ the canonical projection on the "x factor." Furthermore, note that Φ is injective since for all given elements $f, g \in X'$, the equality $f = g$ holds if and only if $f(x) = g(x)$ for all $x \in X$. Now, consider the inverse map $\Phi^{-1} : \Phi(X') \to X'$. We need only prove that for all $x \in X$, the map $y \mapsto \langle \Phi^{-1}(y), x\rangle$ is continuous, but it is trivial since $\langle \Phi^{-1}(y), x\rangle = \pi_x(y)$.

Now, we observe that

$$\begin{aligned}\Phi(\overline{B}) = Y \cap \{y : |\pi_x(y)| \leq \|x\|\,, \pi_{x+x'}(y) = \pi_x(y) + \pi_{x'}(y), \\ \pi_{\lambda\, x}(y) = \lambda\, \pi_x(y) \ \text{ for all } \ x, x' \in X \ \text{ and } \ \lambda \in \mathbb{R}\}\,.\end{aligned}$$

Notice that the set $A_1 = Y \cap \{y : |\pi_x(y)| \leq \|x\| \ \text{ for all } \ x \in X\} = \prod_{x\in X}[-\|x\|\,, \|x\|]$ is compact by Tychonoff's theorem, whilst

$$A_2 = Y \cap \{y : \pi_{x+x'}(y) = \pi_x(y) + \pi_{x'}(y), \pi_{\lambda x}(y) = \lambda\, \pi_x(y) \ \text{ for all } \ x, x' \in X \ \text{ and } \ \lambda \in \mathbb{R}\}$$

is closed as intersection of closed sets. Therefore, we deduce that $\Phi(B) = A_1 \cap A_2$ is compact. □

3.5 Reflexive Spaces

Recall that by the Definition 3.4.1, a Banach space is reflexive if the canonical (isometric) injection $J : X \to X''$ is surjective. The major theorem is the following result of Kakutani.

Theorem 3.5.1 (Kakutani)**.** *Let X be a Banach space. Then, X is reflexive if and only if the unit closed ball $\overline{B} = X \cap \{x : \|x\|_X \leq 1\}$ is compact for the weak topology $\sigma(X, X')$.*

We omit the (rather technical) proof.

Remark 3.5.2. We see that for a reflexive space, the weak $*$ topology is useless. However, for a non-reflexive space that is the dual of a Banach space (as L^∞), the weak $*$ topology furnishes a topology for which the unit ball is compact, which has fundamental applications to calculus of variations and partial differential equations.

We also mention the following theorem that is not trivial, contrary to what one may think.

Theorem 3.5.3. *A Banach space is reflexive if and only if its dual space is reflexive.*

3.6 Separable Spaces

We have the following results.

Theorem 3.6.1. *Let X be a Banach space such that X' is separable. Then, X is separable.*

Remark 3.6.2. L^∞, the dual of L^1, is not separable, although L^1 (as all Lebesgue spaces L^p for $1 \leq p < \infty$) is separable (provided that we consider the space L^1 on an open subset of $\mathbb{R}^d$ for example).

Theorem 3.6.3. *Let X be a Banach space. Then X is reflexive and separable if and only if X' is reflexive and separable.*

We assume the reader familiar with Lebesgue spaces (since they are special cases of Sobolev spaces) and do not recall here the basic results such as the Hölder's inequality (we will see generalisations of it), the convergence theorems of Lebesgue or Fatou, or the inequality for convolutions that will all be treated in the more general setting of Lorentz and Orlicz spaces.

Chapter 4

Convexity

4.1 Introduction

In this chapter, we will restrict to functionals of the form

$$E(u) = \int_\Omega F(\nabla u)dx,$$

where $\Omega \subset \mathbb{R}^d$ is an open subset and $u : \Omega \to \mathbb{R}^n$, while $F : M_{n,d}(\mathbb{R}) \to \mathbb{R}$. The situation between the scalar case ($n = 1$) and the vectorial case ($n > 1$) vastly differ. In this chapter, we will aim to solve the problem

$$\inf_{u \in \mathscr{A}} E(u),$$

where

$$\mathscr{A} = W^{1,p}(\Omega) \cap \{u : u = g \text{ on } \partial\Omega\} \quad 1 \leq p < \infty$$

and $g : \partial\Omega \to \mathbb{R}^n$ is a given boundary data (whose regularity will be fixed later). The model case is $F(X) = |X|^2$, $p = 2$, and $g \in H^{1/2}(\partial\Omega)$. The case $p = 1$ is delicate and we will assume in the rest of the chapter that $p > 1$. We assume further that Ω is a bounded, smooth (Lipschitz regular would be enough) open subset of $\mathbb{R}^d$.

To have a chance to find a minimiser, recalling Theorem 2.1.1, we impose the following coercivity condition

$$F(X) \geq \alpha|X|^p - \beta \qquad \text{for all } X \in M_{n,d}(\mathbb{R}),$$

where $\alpha > 0$ and $\beta \geq 0$ are fixed constants. To apply the proof of Theorem 2.1.1, we need to show the lower semi-continuity of E for the weak convergence in $W^{1,p}$. Indeed, if $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega)$ is a minimising sequence, we deduce in particular that

$$\int_\Omega |\nabla u_k|^p dx \leq \frac{1}{\alpha} F(u_k) + \frac{\beta}{\alpha} \leq \Gamma < \infty.$$

Therefore, if g admits a trace $\widehat{g} : W^{1,p}(\Omega)$, we deduce that $u_k - \widehat{g} \in W_0^{1,p}(\Omega)$, and the Poincaré inequality implies that

$$\begin{aligned}
\|u_k\|_{\mathrm{L}^p(\Omega)} &\leq \|\widehat{g}\|_{\mathrm{L}^p(\Omega)} + \|u_k - \widehat{g}\|_{\mathrm{L}^p(\Omega)} \leq \|\widehat{g}\|_{\mathrm{L}^p(\Omega)} + C_P \|\nabla(u_k - \widehat{g})\|_{\mathrm{L}^p(\Omega)} \\
&\leq \|\widehat{g}\|_{\mathrm{L}^p(\Omega)} + C_P \|\nabla\widehat{g}\|_{\mathrm{L}^p(\Omega)} + C_P \|\nabla u_k\|_{\mathrm{L}^p(\Omega)} \\
&\leq (1 + C_P) \|\widehat{g}\|_{\mathrm{W}^p(\Omega)} + C_P\Gamma < \infty,
\end{aligned}$$

which shows that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$. Therefore, up to a subsequence, we deduce that there exists $u \in W^{1,p}(\Omega)$ such that

$$u_k \underset{k\to\infty}{\rightharpoonup} u \qquad \text{weakly in } W^{1,p}(\Omega).$$

Therefore, we will deduce the existence of a minimiser provided that

$$E(u) \leq \liminf_{k\to\infty} E(u_k) < \infty.$$

If there exists a minimiser, assuming that F is a C^2 function, we deduce that for all $\varphi \in \mathscr{D}(\Omega, \mathbb{R}^n)$, we have $E(u) \leq E(u + t\,\varphi)$, which shows that we must have

$$\frac{d}{dt} E(u + t\,\varphi)_{|t=0} = 0. \tag{4.1.1}$$

We expand

$$E(u + t\,\varphi) = \int_\Omega F(\nabla u + t\nabla\varphi)dx = \int_\Omega F(u)dx + t\int_\Omega \nabla F(\nabla u)\cdot\nabla\varphi\,dx + \frac{t^2}{2}\int_\Omega D^2F(\nabla u)(\nabla\varphi, \nabla\varphi)dx + o(t^2).$$

We deduce from (4.1.1) that

$$\int_\Omega \nabla F(\nabla u)\cdot\nabla\varphi\,dx = 0 \qquad \text{for all } \varphi \in \mathscr{D}(\Omega, \mathbb{R}^n).$$

This shows that u solves in the distributional sense the equation

$$\operatorname{div}\left(\nabla F(\nabla u)\right) = 0 \qquad \text{in } \mathscr{D}'(\Omega). \tag{4.1.2}$$

Furthermore, as the function $t \mapsto E(u + t\,\varphi)$ admits its minimum at $t = 0$, we deduce that

$$\int_\Omega D^2F(\nabla u)(\nabla\varphi, \nabla\varphi)dx = \int_\Omega \sum_{i,j=1}^{d}\sum_{k,l=1}^{n} \frac{\partial^2 F}{\partial p_i^k\,\partial p_j^l}(\nabla u)\frac{\partial\varphi^k}{\partial x_i}\frac{\partial\varphi^l}{\partial x_j}\,dx \geq 0. \tag{4.1.3}$$

4.2 The Scalar Case

First assume that $n = 1$. Then, the inequality (4.1.3) reduces to

$$\int_\Omega \sum_{i,j=1}^{d} \frac{\partial^2 F}{\partial p_i\,\partial p_j}(\nabla u)\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\,dx \geq 0. \tag{4.2.1}$$

This inequality shows that the following condition must hold:

$$\frac{\partial^2 F}{\partial p_i\,\partial p_j}(\nabla u(x))\xi_i\,\xi_j \geq 0 \qquad \text{for all } x \in \Omega,\; \text{for all } \xi \in \mathbb{R}^d. \tag{4.2.2}$$

This inequality shows that F must be a convex function on the domain of F. Recall that a C^2 function F is convex if and only if

$$\xi^t D^2F(p)\xi \geq 0 \qquad \text{for all } p, \xi \in \mathbb{R}^d. \tag{4.2.3}$$

We will show that this condition is a sufficient and necessary condition for the lower semi-continuity of E in the weak topology.

Theorem 4.2.1. *The functional E is lower semi-continuous for the weak convergence in $W^{1,p}(\Omega)$ if and only $F : \mathbb{R}^d \to \mathbb{R}$ is a convex function.*

Proof. Let us first show that the lower semi-continuity implies the convexity of F. Assume for simplicity that $\Omega = Q =]0,1[^d$ (the general argument would follow from a standard covering argument). Let $\varphi \in C_c^\infty(\Omega)$ and $p \in \mathbb{R}^d$. For all $k \in \mathbb{N}$, we divide Q into 2^{kd} cubes of length 2^k denoted by $\{Q_l\}_{l=1}^{2^{kd}}$. Define

$$u_k(x) = \frac{1}{2^k} u\left(2^k(x - x_l)\right) + p \cdot x \qquad \text{for all } x \in Q_l,$$

where x_l is the centre of the cube Q_l and let $u(x) = p \cdot x$. Then, it is easy to see that $u_k \underset{k\to\infty}{\rightharpoonup} u$ in $W^{1,p}(\Omega)$. Therefore, as E is lower semi-continuous, we deduce that

$$\mathscr{L}^d(Q)F(p) = E(u) \leq \liminf_{k\to\infty} E(u_k) = \int_Q F(p + \nabla\varphi)dx.$$

Therefore, the function $u(x) = p \cdot x$ is a minimiser with respect to its own boundary value in ∂Q. This implies that (4.2.2) holds for all $\xi \in \mathbb{R}^d$, and since p was arbitrary, we deduce that F is convex.

Conversely, if F is convex, it is the supremum of affine functions. First assume that

$$F(X) = \max_{1\leq i\leq m} (a_i \cdot X + b_i) \qquad a_i \in \mathbb{R}^d,\ b_i \in \mathbb{R}.$$

Then, we make the decomposition $\Omega = E_1 \cup \cdots \cup E_m$, where

$$E_i = \Omega \cap \{x : F(\nabla u(x)) = a_i \cdot \nabla u(x) + b_i\},$$

and assume without loss of generality that $\mathscr{L}^d(E_i \cap E_j) = 0$ for all $i \neq j$. Then, as the weak convergence implies the convergence of means, we deduce that

$$\begin{aligned}
E(u) = \int_\Omega F(\nabla u)dx &= \sum_{i=1}^m \int_{E_i} (a_i \cdot \nabla u(x) + b_i)\, dx \\
&= \lim_{k\to\infty} \sum_{i=1}^m \int_{E_i} (a_i \cdot \nabla u_k(x) + b_i)\, dx \\
&\leq \liminf_{k\to\infty} \sum_{i=1} \int_{E_i} F(\nabla u_k)dx \\
&\leq \liminf_{k\to\infty} \int_\Omega F(\nabla u_k)dx = \liminf_{k\to\infty} E(u_k),
\end{aligned}$$

where we used that $F(x) \geq a_i \cdot x + b_i$ for all $1 \leq i \leq m$. In general, the result follows thanks to the monotone convergence theorem. □

One of the main goals of the modern theory of calculus of variation is to show how to remedy the lack of coercivity or convexity of functionals to construct critical points of them. In the final chapter of the lecture notes, we will see what kind of methods can be implemented in the case of the area functional.

4.3 The Vectorial Case

We assume that $n \geq 2$ and we consider

$$E(u) = \int_\Omega F(\nabla u)dx,$$

where $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}^n$ and $F : M_{n,d}(\mathbb{R}) \to \mathbb{R}$. We assume as previously that there exists $\alpha > 0$ and $\beta \geq 0$ such that for every matrix $X \in M_{n,d}(\mathbb{R})$, the following inequality holds

$$F(X) \geq \alpha|X|^p - \beta.$$

Once more, we look for a condition that ensures the lower semi-continuity of F for the weak convergence. The condition (4.1.3) becomes

$$\sum_{i,j=1}^{d}\sum_{k,l=1}^{n}\frac{\partial F}{\partial p_i^k\,\partial p_j^l}(\nabla u(x))\eta_k\,\eta_l\,\xi_i\,\xi_j\geq 0$$

for all $x\in\Omega$, $\xi\in\mathbb{R}^d$ and $\eta\in\mathbb{R}^n$. This inequality allows us to introduce the Hadamard-Legendre inequality

$$(\eta\otimes\xi)^t D^2F(X)(\eta\otimes\xi)\geq 0\qquad\text{for all } X\in M_{n,d}(\mathbb{R}),\ \xi\in\mathbb{R}^d,\ \eta\in\mathbb{R}^n,\tag{4.3.1}$$

where $\eta\otimes\xi\in M_{n,d}(\mathbb{R})$ is the matrix whose (k,i) entry is given by $\eta_k\xi_i$ $(1\leq i\leq d,\ 1\leq k\leq n)$. A function satisfying the condition (4.3.1) is called a rank-one convex function. The condition implies that for all X,ξ,η as above, the real variable function

$$f(t)=F(X+t(\eta\otimes\xi))$$

is convex, but it does *not* that F is convex.

If we repeat the proof of Theorem 4.2.1, we arrive as the following condition

$$\mathscr{L}^d(Q)F(X)\leq\int_Q F(X+\nabla\varphi)dx\qquad\text{for all } \varphi\in C_c^\infty(\Omega).$$

It turns out to be the optimal condition to have lower semi-continuity of E for the weak convergence.

Definition 4.3.1. We say that a function $F:M_{n,d}(\mathbb{R})\to\mathbb{R}$ is *quasiconvex* if for all $X\in M_{n,d}(\mathbb{R})$, for all cube $Q\subset\mathbb{R}^d$ and $\varphi\in C_c^\infty(\Omega)$, the following inequality holds:

$$\int_Q F(X)dx\leq\int_Q F(X+\nabla\varphi(x))dx.$$

From now, we also assume that

$$0\leq F(X)\leq C(1+|X|^p).\tag{4.3.2}$$

Theorem 4.3.2. *Assume that F satisfies the growth condition* (4.3.2)*. Then, the functional E is lower semi-continuous for the weak topology if and only if F is quasi-convex.*

We omit the proof.

Checking quasi-convexity is impossible in practice, but another property ensures the lower semi-continuity.

Definition 4.3.3. We say that a function $F:M_{n,d}(\mathbb{R})\to\mathbb{R}$ is poly-convex if for all $X\in M_{n,d}(\mathbb{R})$, $F(X)$ is a convex function of the determinants of minors of X.

Theorem 4.3.4. *A poly-convex function is quasi-convex.*

As a consequence, if

$$F(X)=|X|^p+f(\det X),$$

where $f:\mathbb{R}\to\mathbb{R}$ is a convex function, then E is lower semi-continuous for the weak convergence.

Chapter 5

Plateau's Problem

5.1 Notations

We note $\mathbb{D} = \mathbb{C} \cap \{z : |z| < 1\}$ the unit disk of the plane and $S^1 = \partial\mathbb{D} = \mathbb{C} \cap \{z : |z| = 1\}$ its boundary.

5.2 Statement of the Problem

Let $\gamma : S^1 \to \mathbb{R}^n$ be a continuous injective map and $\Gamma = \gamma(S^1)$. We say that Γ is a Jordan curve. If $u \in W^{1,2}(\mathbb{D}, \mathbb{R}^n)$, its area is given by

$$\mathrm{A}(u) = \int_{\mathbb{D}} |\partial_{x_1} u \wedge \partial_{x_2} u| dx.$$

Recall that if $v = \sum_{i=1}^n v_i\, e_i$ and $w = \sum_{i=1}^n w_i e_i$ (where $(e_1, \cdots, e_n)$ is the canonical base), then

$$v \wedge w = \sum_{1 \leq i < j \leq n} (v_i w_j - v_j w_i)\, e_i \wedge e_j,$$

and the scalar product on $\Lambda^2\mathbb{R}^n$ is given by

$$|v \wedge w|^2 = \sum_{1 \leq i < j \leq n} |v_i w_j - v_j w_i|^2 .$$

In the special case where $n = 3$, we also have

$$\mathrm{A}(u) = \int_{\mathbb{D}} |\partial_{x_1} u \times \partial_{x_2} u| dx,$$

where $\times$ is the vector product. As we mentioned in the introduction, the area is too weak a functional to allow us to minimise it and expect a proper control. Indeed, although we need $u \in W^{1,2}(\mathbb{D})$ to define the area, it does not control the entire gradient in L^2. We will therefore minimise the Dirichlet energy

$$E(u) = \frac{1}{2}\int_{\mathbb{D}} |\nabla u|^2 dx = \frac{1}{2}\int_{\mathbb{D}} \left(\left|\frac{\partial u}{\partial x_1}\right|^2 + \left|\frac{\partial u}{\partial x_2}\right|^2 \right) dx_1\, dx_2$$

amongst conformal maps. The main difficulty is to show that our class is not empty and possesses good compactness properties. Although A is invariant under the entire diffeomorphism group of $\mathbb{D}$, the Dirichlet energy E is also invariant under the group of positive *conformal diffeomorphisms* of the disk: $\mathscr{M}_+(\mathbb{D})$. Although the group is finite-dimensional, it is non-compact and that will create significant technical complications in the proof. Since conformal and holomorphic maps are equivalent, we can fairly easily classify the elements of $\mathscr{M}_+(\mathbb{D})$. On a $f \in \mathscr{M}_+(\mathbb{D})$ if and only if there exists $a \in \mathbb{D}$ and $\theta \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ such that

$$f(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}.$$

Therefore, $\mathscr{M}_+(\mathbb{D})$ is homeomorphic to $\mathbb{D}\times\mathbb{T}$ or equivalently, to $\mathbb{D}\times S^1$. Now, let us prove the conformal invariance of the Dirichlet energy. Using the Cauchy-Riemann operators:

$$\begin{cases} \partial_z = \dfrac{1}{2}\left(\partial_{x_1} - i\,\partial_{x_2}\right) \\ \partial_{\overline{z}} = \dfrac{1}{2}\left(\partial_{x_1} + i\,\partial_{x_2}\right). \end{cases}$$

Therefore, since u is real-valued, we have

$$E(u) = \int_{\mathbb{D}} \left(|\partial_z u|^2 + |\partial_{\overline{z}} u|^2\right)\frac{d\overline{z}\wedge dz}{2i} = 2\int_{\mathbb{D}} |\partial_z u|^2\,\frac{d\overline{z}\wedge dz}{2i}.$$

Therefore, we find that

$$\begin{aligned} E(u\circ f) &= 2\int_{\mathbb{D}} |\partial_z(u\circ f)|^2\frac{d\overline{z}\wedge dz}{2i} = 2\int_{\mathbb{D}} |f'(z)|^2|(\partial_z u)\circ f|^2\frac{d\overline{z}\wedge dz}{2i} \\ &= \int_{\mathbb{D}} |\partial_w u|^2\frac{d\overline{w}\wedge dw}{2i} = E(u). \end{aligned}$$

First, we need to make sure that minimising the Dirichlet energy is equivalent to minimising the area. We have the following result of Morrey.

Theorem 5.2.1 (CITE)**.** *Let $u \in W^{1,2}(\mathbb{D},\mathbb{R}^n)\cap C^0(\overline{\mathbb{D}},\mathbb{R}^n)$. For all $\varepsilon > 0$, there exists a homeomorphism $\Psi : \mathbb{D}\to\mathbb{D}$ such that $\Psi \in W^{1,2}(\mathbb{D},\mathbb{D})$ and furthermore, we have*

$$u\circ\Psi \in W^{1,2}(\mathbb{D},\mathbb{R}^n)\cap C^0(\overline{\mathbb{D}},\mathbb{R}^n)$$

and

$$E(u\circ\Psi) \leq A(u\circ\Psi) + \varepsilon = A(u) + \varepsilon.$$

Another difficulty of the Plateau problem is that we cannot simply expect that a solution $u \in W^{1,2}(\mathbb{D})$ will satisfy $u = \gamma$ on $\partial\mathbb{D}$. We therefore have to introduce a weaker notion of parametrisation for the boundary.

Definition 5.2.2. Let $\Gamma \subset \mathbb{R}^n$ be a Jordan curve and $\gamma : S^1 \to \mathbb{R}^n$ a continuous parametrisation of Γ. We say that a map $\psi : S^1 \to \Gamma$ is weakly monotone if there exists an increasing function $\tau : [0,2\pi]\to\mathbb{R}$ such that $\tau(0) = 0$ and $\tau(2\pi) = 2\pi$ such that

$$\psi\left(e^{i\theta}\right) = \gamma\left(e^{i\tau(\theta)}\right) \quad \text{for all } \theta \in [0,2\pi].$$

The subset of $W^{1,2}(\mathbb{D},\mathbb{R}^n)$ that has the suitable compactness properties is given has follows:

$$\mathscr{P}(\Gamma) = W^{1,2}(\mathbb{D},\mathbb{R}^n)\cap C^0(\overline{\mathbb{D}},\mathbb{R}^n)\cap\left\{u : u_{|\partial\mathbb{D}} \in C^0(\partial\mathbb{D},\Gamma) \text{ and } u \text{ is weakly monotone on } \partial\mathbb{D}\right\}.$$

We note that by trace theory, a weak limit in the class $\mathscr{P}(\Gamma)$ will only be *a prior* in $H^{1/2}(\partial\mathbb{D})$, and the Sobolev injection $H^s(S^1)\hookrightarrow C^0(S^1)$ is only verified for $s > 1/2$. This will be one of the difficulties of the proof.

5.3 A Proof of the Plateau Problems for Rectifiable Curves

5.3.1 On the Length of Curves

If a curve admits a C^1 parametrisation, we can define its length as follows.

Definition 5.3.1. Let $\Gamma \subset \mathbb{R}^n$ be a curve that admits a C^1 parametrisation $\gamma : [a, b] \to \Gamma$. Then, the length of Γ is defined by

$$\mathcal{L}(\Gamma) = \int_0^1 |\gamma'(t)| dt. \tag{5.3.1}$$

For a rectifiable curve (not necessarily C^1), we introduce another definition.

Let $\mathscr{S} = \mathscr{S}([a, b])$ be the set of subdivisions of $[a, b]$, define a function $\mathcal{L}_0 : \mathscr{S} \to \mathbb{R}_+$ such that for every subdivision $\Delta = \{a_0, a_1, \cdots, a_{m-1}\} \in \mathscr{S}$, we have

$$\mathscr{L}_0(\Delta) = \sum_{i=1}^{m-1} |\gamma(a_i) - \gamma(a_{i-1})|.$$

Then, the length of Γ is defined by the following formula

$$\mathscr{L}(\Gamma) = \sup_{\Delta \in \mathscr{S}} \mathscr{L}_0(\Delta). \tag{5.3.2}$$

We now prove the elementary result.

Theorem 5.3.2. *Let $\Gamma \subset \mathbb{R}^n$ be a C^1 curve. Then, we have*

$$\mathscr{L}(\Gamma) = \mathcal{L}(\Gamma).$$

Proof. Let $\gamma : [a, b] \to \Gamma$ be a C^1 parametrisation of .

Étape 1: Let us first prove that $\mathscr{L}(\Gamma) \leq \mathcal{L}(\Gamma)$.

Let $\Delta = \{a_0, a_1, \cdots, a_{m-1}\} \in \mathscr{S}$ be a subdivision of $[a, b]$. Thanks to the fundamental theorem of calculus, we have for all $1 \leq i \leq m-1$:

$$\gamma(a_i) - \gamma(a_{i-1}) = \int_{a_{i-1}}^{a_i} \gamma'(t) dt.$$

The triangle inequality therefore implies that

$$|\gamma(a_i) - \gamma(a_{i-1})| = \left| \int_{a_{i-1}}^{a_i} \gamma'(t) dt \right| \leq \int_{a_{i-1}}^{a_i} |\gamma'(t)| dt.$$

As a consequence, the linearity of the integral shows that

$$\mathscr{L}_0(\Delta) = \sum_{i=1}^{m-1} |\gamma(a_i) - \gamma(a_{i-1})| \leq \sum_{i=1}^{m-1} \int_{a_{i-1}}^{a_i} |\gamma'(t)| dt = \int_a^b |\gamma'(t)| dt = \mathcal{L}(\Gamma).$$

As the inequality is satisfied for every subdivision $\Delta \in \mathscr{S}$, taking the supremum on the left-hand side yields the inequality

$$\mathscr{L}(\Gamma) \leq \mathcal{L}(\Gamma) < \infty.$$

In particular, the left-hand side is a finite quantity!

Étape 2: Let us show now that $\mathcal{L}(\Gamma) \leq \mathscr{L}(\Gamma)$. We need only find a sequence of subdivisions $\{\Delta_m\}_{m \in \mathbb{N}} \subset \mathscr{S}$ such that

$$\mathscr{L}_0(\Delta_m) \underset{m \to \infty}{\longrightarrow} \mathcal{L}(\Gamma).$$

To simplify the notations, assume that $a = 0$ et $b = 1$. Let $m \geq 1$ and for all $0 \leq i \leq m$, define

$$a_i = \frac{i}{m}.$$

The function γ is continuously differentiable, which shows that for all $0 \leq i \leq m-1$, and for all $t \in [\frac{i}{m}, \frac{i+1}{m}]$, we have

$$\gamma(t) = \gamma\left(\frac{i}{m}\right) + \gamma'\left(\frac{i}{m}\right)\left(t - \frac{i}{m}\right) + o\left(t - \frac{i}{m}\right).$$

In particular, we have

$$\gamma\left(\frac{i+1}{m}\right) - \gamma\left(\frac{i}{m}\right) = \frac{1}{m}\gamma'\left(\frac{i}{m}\right) + o\left(\frac{1}{m}\right).$$

Summing those inequalities, we get

$$\mathscr{L}_0(\Delta_m) = \sum_{i=0}^{m-1}\left|\gamma\left(\frac{i+1}{m}\right) - \gamma\left(\frac{i}{m}\right)\right| = \sum_{i=0}^{m-1}\left(\frac{1}{m}\left|\gamma'\left(\frac{i}{m}\right)\right| + o\left(\frac{1}{m}\right)\right) = \frac{1}{m}\sum_{i=0}^{m-1}\left|\gamma'\left(\frac{i}{m}\right)\right| + o(1).$$

Here, we give more details on the last step. As $\gamma \in C^1([a,b])$, we have $C = \sup_{[a,b]}|\gamma'| < \infty$. As a consequence, for all $i \in \{0, \cdots, m-1\}$, we have

$$\begin{aligned}\left|\frac{1}{m}\gamma'\left(\frac{i}{m}\right) + o\left(\frac{1}{m}\right)\right| &= \sqrt{\frac{1}{m^2}\left|\gamma'\left(\frac{i}{m}\right)\right|^2 + \frac{2}{m}\left|\gamma'\left(\frac{i}{m}\right)\right| o\left(\frac{1}{m}\right) + o\left(\frac{1}{m^2}\right)} \\ &= \sqrt{\frac{1}{m^2}\left|\gamma'\left(\frac{i}{m}\right)\right|^2 + o\left(\frac{1}{m^2}\right)} = \frac{1}{m}\left|\gamma'\left(\frac{i}{m}\right)\right| + o\left(\frac{1}{m}\right),\end{aligned}$$

where we used the elementary Taylor expansion

$$\sqrt{1+x} = 1 + \frac{x}{2} + O(x^2).$$

Since the Riemann and Lebesgue integral coincide (in an elementary way) for continuous functions (and continuous functions are Riemann-integrable), we finally deduce that

$$\frac{1}{m}\sum_{i=0}^{m-1}\left|\gamma'\left(\frac{i}{m}\right)\right| \underset{m\to\infty}{\longrightarrow} \int_0^1 |\gamma'(t)|dt,$$

which implies that

$$\lim_{m\to\infty}\mathscr{L}_0(\Delta_m) = \mathcal{L}(\Gamma),$$

and concludes the proof of the theorem. □

Remark 5.3.3. We can also easily show that $\mathscr{L}(\Gamma) = \mathscr{H}^1(\Gamma)$, where $\mathscr{H}^1$ is the 1-dimensional Hausdorff measure [11, **2.10.2, 3.2.46**]

This definition allows us to introduce the notion of rectifiable curve.

Theorem 5.3.4. *We say that a curve $\Gamma \subset \mathbb{R}^n$ is rectifiable if it admits a continuous parametrisation and has finite length:*

$$\mathscr{L}(\Gamma) < \infty.$$

5.3.2 Statement of the Theorem

Theorem 5.3.5 (Douglas, Radó, Courant, Tonelli)**.** *Let $\Gamma \subset \mathbb{R}^n$ be a rectifiable curve. Then, there exists a minimiser u for the Dirichlet energy E in the class $\mathscr{P}(\Gamma)$. Furthermore, any minimiser u of the problem*

$$\inf_{v\in\mathscr{P}(\Gamma)} E(v)$$

is a minimiser of the area functional A in $\mathscr{P}(\Gamma)$, and satisfies the following system of equations:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{D} \\ \langle \partial_z u, \partial_z u \rangle = 0 & \text{in } \mathbb{D}. \end{cases} \tag{5.3.3}$$

Finally, we have $u \in C^\infty(\mathbb{D}, \mathbb{R}^n) \cap C^0(\overline{\mathbb{D}}, \mathbb{R}^n)$ and the restriction $u_{\partial\mathbb{D}} : \partial\mathbb{D} \to \mathbb{R}^n$ is an homeomorphism *of S^1 into Γ.*

Since harmonic functions are real parts of holomorphic functions in dimension 2, if $u(z) = \mathrm{Re}\,(f(z))$, with f holomorphic, then we get

$$\partial_z u = \frac{1}{2} f'(z),$$

which yields

$$|\nabla u|^2 = 4|\partial_z u|^2 = |f'(z)|^2.$$

Therefore, the maximum principle implies that f' has finitely many zeroes (since u cannot be a constant function), which shows that u is an immersion outside finitely many points, called *branch points.* If $n \geq 4$, branch points are unavoidable. For example, if $\gamma : [0, 2\pi] \to \mathbb{C}^2 \simeq \mathbb{R}^4, \theta \to (e^{2i\theta}, e^{3i\theta})$ and $\Gamma = \gamma([0, 2\pi])$, then the immersion $u : \mathbb{D} \to \mathbb{C}^2$ given by

$$u(z) = (z^2, z^3) \quad \text{for all } z \in \mathbb{D}$$

is a minimal immersion that solves the Plateau problem for Γ, but it is an isolated (and unique) branch point at $z = 0$. However, in dimension 3 ($n = 3$), Osserman showed that there are no interior branch points (see also Gulliver–Osserman–Royden). Before establishing existence, let us prove a regularity result on the weak limits in the class $\mathscr{P}(\Gamma)$.

5.3.3 Properties of Minimising Sequences

Proposition 5.3.6. *Let u be a weak limit in $W^{1,2}$ of a minimising sequence of E in $\mathscr{P}(\Gamma)$. Then, u is a harmonic function, i.e., it satisfies the Laplace equation:*

$$\Delta u = 0 \qquad \text{in } \mathbb{D}.$$

Proof. The proof is an application of the ideas of the Euler–Lagrange equation. Let $\varphi \in C_c^\infty(\mathbb{D}, \mathbb{R}^n)$ be a test function. If $\{u_k\}_{k\in\mathbb{N}}$ is a minimising sequence of E in the class $\mathbb{P}(\Gamma)$, then, we have $u_k + t\,\varphi \in \mathbb{P}(\Gamma)$ for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$. In particular, we have

$$\inf_{v\in\mathbb{P}(\Gamma)} E(v) = \lim_{k\to\infty} E(u_k) \leq \liminf_{k\to\infty} E(u_n + t\,\varphi).$$

Now, we have

$$E(u_k + t\varphi) = E(u_k) + t\int_{\mathbb{D}} \nabla u_k \cdot D\varphi\, dx + t^2 E(\varphi).$$

Since $u_k \hookrightarrow u_\infty$ weakly in $W^{1,2}$ as $k \to \infty$, we deduce that

$$\int_{\mathbb{D}} \nabla u_k \cdot \nabla\varphi\, dx \underset{k\to\infty}{\longrightarrow} \int_{\mathbb{D}} \nabla u \cdot \nabla\varphi\, dx.$$

Therefore, we deduce by lower semi-continuity of the Dirichlet energy that for all $t \in \mathbb{R}$

$$\inf_{v\in\mathbb{P}(\Gamma)} E(v) \leq E(u) + t\int_{\mathbb{D}} \nabla u \cdot \nabla\varphi\, dx + \frac{t^2}{2}\int_{\mathbb{D}} |\nabla\varphi|^2 dx.$$

Therefore, we deduce that

$$\int_{\mathbb{D}} \nabla u \cdot \nabla\varphi\, dx = 0,$$

which shows that u solves the Laplace equation in the distributional sense, and concludes the proof of the proposition. ☐

We see that variations in the target give us the first equation in the system (5.3.3). The second equation will be given by variations in the domain, and this equation corresponds to the stationarity condition. It can be stated as follows: for all $X \in C_c^\infty(\mathbb{D}, \mathbb{R}^2)$, we have

$$\frac{d}{dt} E\left(u \circ (\mathrm{Id} + t\, X)\right)_{|t=0} = 0. \tag{5.3.4}$$

Proposition 5.3.7. *A map $u : W^{1,2}(\mathbb{D}, \mathbb{R}^n)$ satisfies the stationarity condition if and only its Hopf differential, given by*

$$\begin{aligned} h_0(u) &= \langle \partial u, \partial u \rangle = \langle \partial_z u, \partial_z u \rangle dz^2 \\ &= \frac{1}{4}\left(|\partial_x u|^2 - |\partial_y u|^2 - 2i \langle \partial_x u, \partial_y u \rangle \right) dz^2, \end{aligned}$$

is holomorphic.

Remark 5.3.8.

1. In particular, the associated equation is given by

$$\overline{\partial} h_0(u) = 0,$$

where $\overline{\partial} = \partial_{\overline{z}} d\overline{z} = \frac{1}{2}\left(\partial_x + i\, \partial_y\right)\left(dx - i\, dy\right)$ is the Cauchy-Riemann operator.

2. The Hopf differential is an example of a holomorphic quadratic differential. Since the underlying Riemann surface is a disk, it simply corresponds to a holomorphic function on the disk. Alternatively ([10, **11.3** p. 308]), a quadratic differential on a Riemann surface can be described as a section of a the symmetric holomorphic bundle. Here, we can either see the objects formally, or as a family $(U_i, \varphi_i, f_i)_{i \in I}$ of injective holomorphic maps $\varphi_i : U_i \to \mathbb{C}$ and holomorphic functions $f_i : \varphi_i(U_i) \to \mathbb{C}$ such that for all $i, j \in I$ such that $U_i \cap U_j \neq 0$, if $\psi = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$

$$f_j(z)\psi''(z) = f_i(z). \tag{5.3.5}$$

Notice that by the definition of Riemann surfaces, the transition map ψ is a holomorphic map.

Proof. Let $\{x_t\}_{t \geq 0}$ be the flow associated to a fixed vector field $X \in C_c^\infty(\mathbb{D}, \mathbb{R}^2)$. It satisfies the equation

$$\begin{cases} \dfrac{d}{dt} x_t(x) = X(x_t(x)) & \text{for all } t > 0 \\ x_0(x) = x \end{cases}$$

A unique solution exists thanks to a standard application of the Cauchy-Lipschitz theorem. The chain rule shows that

$$\partial_{x_i} u(x_t) = \sum_{j=1}^{2} \partial_{x_j} u(x_t) \partial_{x_i} x_t^i.$$

In particular, we deduce that

$$\int_{\mathbb{D}} |\nabla(u(x_t))|^2 dx = \int_{\mathbb{D}} |\nabla u|^2 dx + 2t \int_{\mathbb{D}} \sum_{i,j=1}^{2} (\partial_{x_i} u)(x_t) \cdot (\partial_{x_j} u)(x_t) \partial_{x_i} X^j + o(t).$$

Since X is compactly supported in the disk $\mathbb{D}$, we have for all $f \in L^1(\mathbb{D})$ and $\varphi \in C^\infty(\mathbb{D})$

$$\begin{aligned} \int_{\mathbb{D}} f(x_t)\varphi(x) dx &= \int_{x_t(\mathbb{D})} f(y)\varphi(x_t^{-1}) d(x_t^{-1}(y)) \\ &= \int_{\mathbb{D}} f(y)\left(\varphi(y) - t\nabla\varphi \cdot X + o(t)\right)(1 - t\, \mathrm{div}\, X + o(t)) dy \\ &= -t \int_{\mathbb{D}} f(y)\left(\nabla\varphi \cdot X + \varphi\, \mathrm{div}\, X\right) dy + o(t). \end{aligned}$$

We deduce that

$$\frac{d}{dt}\left(\int_{\mathbb{D}} f(x_t)\varphi(x)dx\right)_{|t=0} = -\int_{\mathbb{D}} f\ \mathrm{div}(\varphi\, X)dx. \tag{5.3.6}$$

Therefore, we deduce that

$$\frac{d}{dt}\left(\int_{\mathbb{D}} |\nabla(u(x_t))|^2 dx\right)_{|t=0} = -\int_{\mathbb{D}} |\nabla u|^2\,\mathrm{div}\, X\, dx + 2\int_{\mathbb{D}} \sum_{i,j=1}^{2} \partial_{x_i} u\cdot \partial_{x_j} u\, \partial_{x_i} X^j\, dx,$$

that we can rewrite (since the equation is verified for all X as above)

$$\frac{\partial}{\partial x_i}\left(|\nabla u|^2\right) - 2\sum_{j=1}\frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_i}\cdot\frac{\partial u}{\partial x_j}\right) = 0 \quad \text{for all } 1\leq i\leq 2.$$

Finally, the equation can be rewritten as

$$\begin{cases} \dfrac{\partial}{\partial x_1}\left(\left|\dfrac{\partial u}{\partial x_1}\right|^2 - \left|\dfrac{\partial u}{\partial x_2}\right|^2\right) + 2\,\dfrac{\partial}{\partial x_2}\left(\dfrac{\partial u}{\partial x_1}\cdot\dfrac{\partial u}{\partial x_2}\right) = 0 & \text{in } \mathbb{D}\\ \dfrac{\partial}{\partial x_2}\left(\left|\dfrac{\partial u}{\partial x_1}\right|^2 - \left|\dfrac{\partial u}{\partial x_2}\right|^2\right) - 2\,\dfrac{\partial}{\partial x_1}\left(\dfrac{\partial u}{\partial x_1}\cdot\dfrac{\partial u}{\partial x_2}\right) = 0 & \text{in } \mathbb{D}\end{cases}$$

and we recognise the Cauchy-Riemann equations, which concludes the proof of the proposition. □

We need to strengthen this result in the case of the Plateau problem since we want to show that the holomorphic function $h_0(u)$ vanishes identically.

Proposition 5.3.9. *Let $u\in W^{1,2}(\mathbb{D},\mathbb{R}^n)$ be such that*

$$\frac{d}{dt}E\left(u\circ(\mathrm{Id}+t\,X)\right)_{|t=0} = 0$$

pour tout $X\in C^{\infty}(\overline{\mathbb{D}},\mathbb{R}^2)$ such that $X(\cos(\theta),\sin(\theta))\cdot(\cos(\theta),\sin(\theta)) = 0$ pour tout $\theta\in[0,2\pi]$. Then, the Hopf differential vanishes identically, i.e.,

$$|\partial_{x_1}u|^2 - |\partial_{x_2}u|^2 - 2i\langle \partial_{x_1}u, \partial_{x_2}u\rangle = 0.$$

Remark 5.3.10. The major difference is that we do not assume that the family of vector-fields has compact support.

Remark 5.3.11. The boundary condition implies that X preserves $\mathbb{D}$ and as a consequence, (5.3.6) still holds. Therefore, the stationarity condition implies that

$$\lim_{r\to 1}\int_{B(0,r)}\left(-|\nabla u|^2\,\mathrm{div}\,X + 2\sum_{i,j=1}^{2}\partial_{x_i}u\cdot\partial_{x_j}u\,\partial_{x_i}X^j\right)dx = 0.$$

Using complex notations $X = X^1 + i\,X^2$, the equation becomes

$$\lim_{r\to 1}\int_{B(0,r)}\mathrm{Re}\left(h_0(u)\frac{\partial X}{\partial\overline{z}}\right)dx = 0,$$

or

$$\lim_{r\to 1}\mathrm{Re}\left(\int_{B(0,r)} h_0(u)\frac{\partial X}{\partial\overline{z}}\frac{d\overline{z}\wedge dz}{2i}\right) = 0.$$

Integrating by parts and using the holomorphic of h_0 ($\partial_{\overline{z}}h_0(u) = 0$), we have by Stokes theorem

$$\int_{B(0,r)} h_0(u)\frac{\partial X}{\partial\overline{z}}\frac{d\overline{z}\wedge dz}{2i} = \frac{1}{2i}\int_{B(0,r)}\frac{\partial}{\partial\overline{z}}\left(h_0(u)X\right)d\overline{z}\wedge dz$$

$$= \frac{1}{2i}\int_{B(0,r)} d\left(h_0(u)X dz\right) = \frac{1}{2i}\int_{\partial B(0,r)} h_0(u)X\, dz.$$

Finally, we obtain the equation

$$\lim_{r\to 0}\mathrm{Re}\,\left(\frac{1}{2\pi i}\int_{\partial B(0,r)} h_0(u)X\, dz\right) = 0. \tag{5.3.7}$$

We take $X(z) = if(z)\,z$ where f is an arbitrary real function that only depends on the angle θ in a neighbourhood of $\partial\mathbb{D}$. Notice that if $\iota : \partial B(0,r) \to \mathbb{R}^2$ is the standard inclusion, then $\iota^*(dz) = i\,z\,d\theta$ in polar coordinates $z = \rho\, e^{i\theta}$. This follows from the fact that $\rho = r$ is constant, which shows that

$$\iota^*(dz) = d(re^{i\theta}) = i\, re^{i\theta} d\theta = i\, z\, d\theta.$$

Therefore, (5.3.7) becomes

$$\lim_{r\to 0}\mathrm{Re}\,\left(\frac{1}{2\pi i}\int_0^{2\pi} h_0(u)(r\,z)f(\theta)\,r^2z^2\,d\theta\right) = \lim_{r\to 0}\mathrm{Re}\,\left(\frac{1}{2\pi i}\int_0^{2\pi} h_0(u)(r\,e^{i\theta})f(\theta)\,r^2e^{2i\theta}\,d\theta\right).$$

Now, recall the Poisson formula: for all harmonic map $u : \mathbb{D} \to \mathbb{R}^n$ such that $u_{\partial\mathbb{D}} \in C^0(\partial\mathbb{D})$, we have

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}u(e^{i\theta})d\theta.$$

Therefore, we fix $z_0 \in \mathbb{D}$ and we let f above be given by the Poisson potential

$$f(\theta) = \frac{1-|z_0|^2}{|e^{i\theta}-z_0|^2}.$$

For all $0 < r < 1$, the function function $g(z) = h_0(u)(z)(z)^2$ is holomorphic, and therefore harmonic, which implies that

$$g(r\,z_0) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z_0|^2}{|e^{i\theta}-z_0|^2}h_0(u)(r\,e^{i\theta})r^2e^{2i\theta}\,d\theta.$$

Therefore, Hopf condition (5.3.7) finally gives

$$\mathrm{Im}\,(g(z_0)) = \lim_{r\to 1}\mathrm{Re}\,(-ig(r\,z_0)) = \lim_{r\to 1}\mathrm{Re}\,\left(-\frac{1}{2\pi i}\int_0^{2\pi}\frac{1-|z_0|^2}{|e^{i\theta}-z_0|^2}h_0(u)(r\,e^{i\theta})r^2e^{2i\theta}\,d\theta\right) = 0.$$

Since the condition is satisfied for all $z_0 \in \mathbb{D}$, we deduce that the imaginary part of the holomorphic function g vanishes identically, but by the maximum principle, this implies that g vanishes identically. In particular, h_0 vanishes on $\mathbb{D}\setminus\{0\}$, and therefore vanishes identically.

5.3.4 Refining the Plateau Class

The next issue is caused by the conformal invariance of the Dirichlet energy. Indeed, recall that the conformal group of the disk $\mathscr{M}_+(\mathbb{D})$ (also known as the Möbius group) is homeomorphic to the *non-compact space* $\mathbb{D}\times S^1$. Explicitly, recall that we can parametrise it by as follows:

$$f_{a,\theta}(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z} \qquad |a|<1,\ 0\leq\theta\leq 2\pi.$$

In particular, taking $\theta = 0$ and choosing a sequence $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{D}$ such that

$$\lim_{k\to\infty} a_n = 1,$$

the sequence of conformal maps $f_{a_n,0}$ converges pointwise to a constant map! In particular, if $\{u_k\}_{k\in\mathbb{N}}$ is a minimising sequence, then up to a subsequence, $\{u_k\}_{k\in\mathbb{N}}$ converges weakly to a function $u_\infty \in W^{1,2}(\mathbb{D})$.

Furthermore, up to extracting another subsequence, we can assume that $\{u_k\}_{k\in\mathbb{N}}$ converges almost everywhere to u_∞. However, the sequence $\{u_k \circ f_k\}_{k\in\mathbb{N}}$ converges almost everywhere to a constant function, which shows that its weak limit cannot solve the Plateau problem. Therefore, we will need to impose a further restriction that guarantees compactness of the sequence. Such a condition is known under the name of *three-point normalisation.*

Why three? The Möbius group happens to be 3-transitive, as one can see in the following elementary lemma.

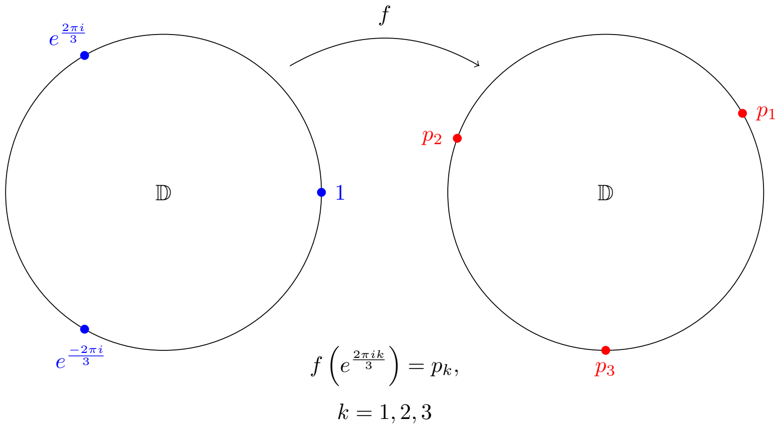


Figure 5.1: 3-transitivity of the Möbius group

Lemma 5.3.12 (3-transitivity of the Möbius group)**.** *Let $p_1, p_2, p_3 \in \partial\mathbb{D}$ be three distinct points ordered in positive trigonometric order. Then, there exists a unique conformal map $f \in \mathscr{M}_+(\mathbb{D})$ such that*

$$f\left(e^{\frac{2\pi i\, k}{3}}\right) = p_k \qquad \textit{for all } k = 1, 2, 3.$$

The proof will be done in the exercise sessions. Thanks to this lemma, we can define a new class for the Plateau problem. Let $\gamma : [0, 2\pi] \to \Gamma$ be a positive parametrisation of Γ (namely, whose orientation coincides with the one taken in the definition of the Plateau class $\mathscr{P}(\Gamma)$) and $q_1, q_2, q_3 \in \Gamma$ a monotone sequence of points (such that $q_k = \gamma(\theta_k)$ with $\theta_1 < \theta_2 < \theta_k$). If p_1, p_2, p_3 are any arbitrary three points ordered by increasing trigonometric order on $\partial\mathbb{D}$ (for example, we can take $p_k = e^{\frac{2\pi i k}{3}}$, $k = 1, 2, 3$), we define a subclass of $\mathscr{P}(\Gamma)$ as follows:

$$\mathscr{P}^*(\Gamma) = \mathscr{P}(\Gamma) \cap \{u : u(p_k) = q_k \text{ for all } k = 1, 2, 3\}.$$

Lemma 5.3.12 shows that

$$\inf_{u\in\mathscr{P}^*(\Gamma)} E(u) = \inf_{u\in\mathscr{P}(\Gamma)} E(u).$$

The main goal now is to prove the closure of $\mathscr{P}^*(\Gamma)$ for the sequential weak topology on $W^{1,2}(\mathbb{D})$. This result is contained in the following theorem.

5.3.5 Weak Closure of the Plateau Class

Theorem 5.3.13. *For all*

$$\inf E(\mathscr{P}^*(\Gamma)) \leq C < \infty,$$

the trace on $\partial\mathbb{D}$ of elements $u \in \mathscr{P}^*(\Gamma)$ such that $E(u) \leq C$ is equicontinuous.

In particular, using the Arzelà-Ascoli theorem, we will be able to extract a subsequence converging strongly on the boundary to a continuous function. In particular, we get the following corollary.

Corollary 5.3.14. *Let $\{u_k\}_{k\in\mathbb{N}} \subset \mathscr{P}^*(\Gamma)$ be a sequence that weakly converges to a map $u_\infty \in W^{1,2}(\mathbb{D})$ for the weak topology. Then, the restriction $(u_\infty)_{|\partial\mathbb{D}} \in H^{\frac{1}{2}}(S^1)$ is a continuous and monotone function, i.e., $(u_\infty)_{|\partial\mathbb{D}} \in C^0(\partial\mathbb{D})$ and there exists an increasing function $\tau : [0, 2\pi] \to \mathbb{R}$ such that $\tau(0) = 0$, $\tau(2\pi) = 2\pi$, and $u_\infty(e^{i\theta}) = \gamma\left(e^{i\tau(\theta)}\right)$.*

The theorem is based on a fundamental lemma due to Courant, and that has had since a major influence on the entire field of calculus of variation (the idea of extracting a "good slice" is far-reaching).

Lemma 5.3.15 (Courant-Lebesgue Lemma)**.** *Let $u \in W^{1,2}(\mathbb{D}, \mathbb{R}^n)$ and let $a \in \partial\mathbb{D}$. Then, for all $0 < \delta < 1$, there exists $\rho \in [\delta, \sqrt{\delta}]$ such that*

$$\nabla u \in L^2(\partial B(a,\rho) \cap \mathbb{D})$$

and furthermore, we have for almost all $x, y \in \partial B(a,\rho) \cap \mathbb{D}$ the inequalities

$$|u(x) - u(y)|^2 \leq \left(\int_{\partial B(a,\rho)\cap\mathbb{D}} |\nabla u| d\mathscr{H}^1\right)^2 \leq \frac{4\pi}{\log\left(\frac{1}{\delta}\right)} \int_{\mathbb{D}} |\nabla u|^2 dx.$$

Proof. The logarithm indicates us how to prove the inequality (notice that the first one follows from the Sobolev embedding $W^{1,1}(I) \hookrightarrow C^0(I)$ for all interval $I \subset \mathbb{R}$). We have by the co-area formula

$$\begin{aligned}
\int_{\mathbb{D}\cap B_{\sqrt{\delta}}\setminus\overline{B}_\delta(a)} |\nabla u|^2 dx &= \int_\delta^{\sqrt{\delta}} \left(\int_{\mathbb{D}\cap\cap\partial B(a,r)} |\nabla u|^2 d\mathscr{H}^1\right) dr = \int_\delta^{\sqrt{\delta}} \left(r\int_{\mathbb{D}\cap\partial B(a,r)} |\nabla u|^2 d\mathscr{H}^1\right) \frac{dr}{r} \\
&\geq \left(\int_\delta^{\sqrt{\delta}} \frac{dr}{r}\right) \inf_{\delta\leq r\sqrt{\delta}} \left(r\int_{\mathbb{D}\cap\partial B(a,r)} |\nabla u|^2 d\mathscr{H}^1\right) \\
&= \frac{1}{2}\log\left(\frac{1}{\delta}\right) \inf_{\delta\leq r\sqrt{\delta}} \left(r\int_{\mathbb{D}\cap\partial B(a,r)} |\nabla u|^2 d\mathscr{H}^1\right).
\end{aligned}$$

In particular, there exists $\rho \in [\delta, \sqrt{\delta}]$ such that

$$\rho\int_{\mathbb{D}\cap\partial B(a,\rho)} |\nabla u|^2 d\mathscr{H}^1 \leq \frac{2}{\log\left(\frac{1}{\delta}\right)} \int_{\mathbb{D}\cap B_{\sqrt{\delta}}\setminus\overline{B}_\delta(a)} |\nabla u|^2 dx. \tag{5.3.8}$$

Now, the Cauchy-Schwarz inequality implies that

$$\begin{aligned}
\int_{\mathbb{D}\cap\partial B(a,\rho)} |\nabla u| d\mathscr{H}^1 &\leq \sqrt{\mathscr{H}^1\left(\mathbb{D}\cap\partial B(a,\rho)\right)}\sqrt{\int_{\mathbb{D}\cap\partial B(a,\rho)} |\nabla u|^2 d\mathscr{H}^1} \\
&= \sqrt{2\pi}\sqrt{\rho\int_{\mathbb{D}\cap\cap\partial B(a,\rho)} |\nabla u|^2 d\mathscr{H}^1}.
\end{aligned} \tag{5.3.9}$$

Thanks to (5.3.8) and (5.3.9), we deduce that

$$\begin{aligned}
\left(\int_{\mathbb{D}\cap\partial B(a,\rho)} |\nabla u| d\mathscr{H}^1\right)^2 &\leq \frac{4\pi}{\log\left(\frac{1}{\delta}\right)} \int_{\mathbb{D}\cap B_{\sqrt{\delta}}\setminus\overline{B}_\delta(a)} |\nabla u|^2 dx \\
&\leq \frac{4\pi}{\log\left(\frac{1}{\delta}\right)} \int_{\mathbb{D}} |\nabla u|^2 dx.
\end{aligned} \tag{5.3.10}$$

Finally, the result follows by the Sobolev embedding $W^{1,1}(I) \hookrightarrow C^0(I)$ (for all interval $I \subset \mathbb{R}$). Indeed, for all $x, y \in I$, assuming that $v \in C^\infty(I)$, recall that we have

$$v(x) - v(y) = \int_y^x v'(t)dt,$$

which implies that

$$|v(x) - v(y)| \le \int_y^x |v'(t)|dt,$$

and by density, this inequality holds for all $v \in W^{1,1}(I)$. Therefore, for all $p, q \in \mathbb{D} \cap \partial B(0, \rho)$

$$|u(p) - u(q)| \le \int_{\mathbb{D}\cap\partial B(a,\rho)} \left|\frac{1}{r}\partial_\theta u\right| d\mathscr{H}^1 \le \int_{\mathbb{D}\cap\partial B(a,\rho)} |\nabla u| d\mathscr{H}^1 \tag{5.3.11}$$

since

$$|\nabla u|^2 = |\partial_r u|^2 + \frac{1}{r^2}|\partial_\theta u|^2 .$$

The inequality is therefore proven by combining (5.3.10) and (5.3.11). □

We can now move to the proof of Theorem 5.3.13.

Proof. (**of Theorem 5.3.13**) Define

$$\mathscr{P}^*_C(\Gamma) = \mathscr{P}^*(\Gamma) \cap \{u : E(u) \le C\} .$$

Recall that a family of continuous functions is equicontinuous provided that there exists a uniform modulus of continuity, namely:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \forall u \in \mathscr{P}^*_C(\Gamma),\ \forall p, q \in \partial\mathbb{D},\ |p - q| < \delta \implies |u(p) - u(q)| < \varepsilon.$$

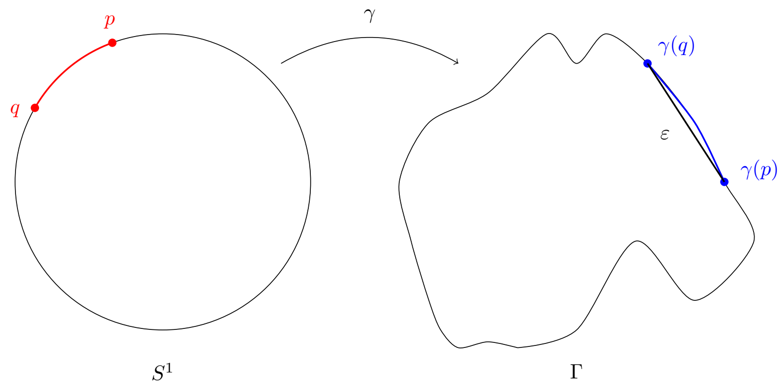
Recall that Γ admits a injective, continuous parametrisation $\gamma : S^1 \to \Gamma$. This property will allow us to show the following reverse equi-continuity that we state as follows.

Lemma 5.3.16. *Let Γ be a Jordan curve and $\gamma : S^1 \to \Gamma$ be a continuous, injective parametrisation. Then, the following property is verified:*

$$\forall\, \varepsilon > 0,\ \exists\, \eta > 0 \text{ such that } \forall\, 0 < \theta_1 < \theta_2 \le 2\pi,$$

$$\left|\gamma\left(e^{i\theta_1}\right) - \gamma\left(e^{i\theta_2}\right)\right| < \eta \implies \min\left\{\sup_{\theta\in]0,\theta_1]\cup[\theta_2,2\pi]} \left|\gamma\left(e^{i\theta}\right) - \gamma\left(e^{i\theta_1}\right)\right|, \sup_{\theta_1\le\theta\le\theta_2} \left|\gamma\left(e^{i\theta}\right) - \gamma\left(e^{i\theta_1}\right)\right|\right\} < \varepsilon. \tag{5.3.12}$$

Geometrically, this condition can be be understood as follows: for all $\varepsilon > 0$ any two points $p, q \in S^1$, if $\gamma(p)$ and $\gamma(q)$ are contained in a sufficiently small ball, the image of the shortest arc on S^1 joining p and q will be included in the ball $B(\gamma(p), \varepsilon) \subset \mathbb{R}^n$.



Notice that in this example, $\eta = \varepsilon$.

Due to periodicity, the condition would not be satisfied if we only took the second term in the minimum, as one can see by taking $\theta_2 = 2\pi$, $\theta = \pi$ and a sequence $\left\{\theta_1^k\right\}_{k\in\mathbb{N}} \subset]0, \pi]$ such that $\theta_1^k \to 0$ as $k \to \infty$.

Proof. (of Lemma 5.3.16 Notice that the map $\gamma :]0,2\pi] \to \mathbb{R}^2, \theta \mapsto \gamma\left(e^{i\theta}\right)$ is an injective and continuous map. Assume by contradiction that (5.3.12) is not satisfied. Then, there exists $\varepsilon_0 > 0$ such that for all $\eta > 0$, there exists $0 < \theta_1 < \theta_2 \leq 2\pi$ such that

$$\begin{cases} \left|\gamma\left(e^{i\theta_1}\right) - \gamma\left(e^{i\theta_2}\right)\right| < \eta \quad \text{and there exists } \theta \in [\theta_1, \theta_2] \text{ and } \widetilde{\theta} \in]0, \theta_1] \cup [\theta_2, 2\pi] \text{ such that} \\ \left|\gamma\left(e^{i\theta}\right) - \gamma\left(e^{i\theta_1}\right)\right| \geq \varepsilon_0 \qquad \text{and} \qquad \left|\gamma\left(e^{i\widetilde{\theta}}\right) - \gamma\left(e^{i\theta_1}\right)\right| \geq \varepsilon_0. \end{cases}$$

For simplicity of notations, we write $\gamma(\theta)$ instead of $\gamma\left(e^{i\theta}\right)$ from now on. Therefore, we obtain sequences $\{\theta_1^k\}_{k\in\mathbb{N}}, \{\theta^k\}_{k\in\mathbb{N}}, \left\{\widetilde{\theta}^k\right\}_{k\in\mathbb{N}}, \{\theta_2^k\}_{k\in\mathbb{N}} \subset]0,2\pi]$ such that $\theta_1^k \leq \theta^k \leq \theta_2^k$ and $\widetilde{\theta}^k \in]0, \theta_1^k] \cup [\theta_2^k, 2\pi]$ for all $k \in \mathbb{N}$ and

$$\left|\gamma\left(\theta_1^k\right) - \gamma\left(\theta_2^k\right)\right| \leq \frac{1}{k+1} \qquad \text{and} \qquad \left|\gamma\left(\theta^k\right) - \gamma\left(\theta_1^k\right)\right| \geq \varepsilon_0 > 0.$$

By compactness of the interval $[0, 2\pi]$, we deduce that up to a subsequence, we have

$$\theta_1^k \underset{k\to\infty}{\longrightarrow} \theta_1^\infty \in [0,2\pi], \qquad \theta_2^k \underset{k\to\infty}{\longrightarrow} \theta_2^\infty \in [0,2\pi], \qquad \theta^k \underset{k\to\infty}{\longrightarrow} \theta^\infty \in [0,2\pi] \qquad \text{and} \qquad \widetilde{\theta}^k \underset{k\to\infty}{\longrightarrow} \widetilde{\theta}^\infty \in [0,2\pi].$$

Furthermore, we have $\theta_1^\infty \leq \theta^\infty \leq \theta_2^\infty$ and either $0 \leq \widetilde{\theta}^\infty \leq \theta_1^\infty$ or $\theta_2^\infty \leq \widetilde{\theta}^\infty \leq 2\pi$. Finally, by continuity of γ, we have

$$|\gamma(\theta_1^\infty) - \gamma(\theta_2^\infty)| \leq 0 \qquad \text{and} \qquad |\gamma(\theta^\infty) - \gamma(\theta_1^\infty)| \geq \varepsilon_0 > 0 \qquad \text{and} \qquad \left|\gamma\left(\widetilde{\theta}^\infty\right) - \gamma\left(\widetilde{\theta}_1^\infty\right)\right| \geq \varepsilon_0 > 0.$$

We distinguish two cases: if $\theta_1^\infty > 0$, by injectivity of γ on $]0, 2\pi]$, we deduce that $\theta_1^\infty = \theta_2^\infty$, but this implies that $\theta^\infty = \theta_1^\infty = \theta_2^\infty$ and the inequality

$$|\gamma(\theta^\infty) - \gamma(\theta_1^\infty)| \geq \varepsilon_0 > 0$$

is absurd. If $\theta_1^\infty = 0$, then we deduce that $\theta_1^\infty = 2\pi$, which implies that $\widetilde{\theta}^\infty = 0$ or $\widetilde{\theta}^\infty = 2\pi$. In both cases, we have $\gamma(\widetilde{\theta}^\infty) = \gamma(\theta_1^\infty) = \gamma(0)$ and once more, the inequality

$$|\gamma(\theta^\infty) - \gamma(\theta_1^\infty)| \geq \varepsilon_0 > 0$$

is absurd. Therefore, the claim is proved. □

We can now return to the proof of the theorem. Recall that we fixed three points $p_1, p_2, p_3 \in \partial\nabla$ and $q_1, q_2, q_3 \in \Gamma$ ordered by positive trigonometric order. Let $\varepsilon > 0$ be such that

$$2\,\varepsilon < \min\{|q_1 - q_2|, |q_1 - q_3|, |q_2 - q_3|\} \tag{5.3.13}$$

and $\delta > 0$ (to be fixed later) be such that

$$2\sqrt{\delta} < \min\{|p_1 - p_2|, |p_1 - p_3|, |p_2 - p_3|\}\,. \tag{5.3.14}$$

For all $p, q \in \partial\mathbb{D}$ such that $|p - q| < \delta$ and $a \in \partial\mathbb{D}$ be the middle point on the (smallest) arc joining p and q such that $|p - a| = |q - a| < \dfrac{\delta}{2}$. Thanks to the Courant-Lesbegue lemma (Lemma 5.3.15), there exists $\rho \in [\delta, \sqrt{\delta}]$ be such that

$$\sup_{x,y\in\mathbb{D}\cap\partial B(a,\rho)} |u(x) - u(y)| \leq \sqrt{\frac{4\pi}{\log\left(\frac{1}{\delta}\right)}\int_{\mathbb{D}} |\nabla u|^2 dx} \leq \sqrt{\frac{4\pi C}{\log\left(\frac{1}{\delta}\right)}}.$$

Since u is continuous up to the boundary, we deduce that if p' and q' are the two points given by the intersection of $\partial\mathbb{D}$ and $\partial B(a, \rho)$, then

$$|u(p') - u(q')| \leq \sqrt{\frac{4\pi C}{\log\left(\frac{1}{\delta}\right)}}$$

We now fix $\delta > 0$ small enough such that

$$\sqrt{\frac{4\pi C}{\log\left(\frac{1}{\delta}\right)}} < \eta,$$

which yields

$$0 < \delta < e^{-\frac{4\pi C}{\eta^2}},$$

then by Lemma 5.3.16, the smallest arc—let us denote it by C—joining $u(p')$ and $u(q')$ is contained in $B_{\mathbb{R}^n}(u(p'), \varepsilon)$. Furthermore, the condition on $\varepsilon > 0$ ensures that C contains at most one point q_i $(1 \leq i \leq 3)$. Furthermore, $\partial\mathbb{D} \cap B(a, \sqrt{\delta})$ contains at most one point p_j $(1 \leq j \leq 3)$, so by weak monotonocity of u on $\partial\mathbb{D}$, the arc C coincides with $u(\partial\mathbb{D} \cap B(a, \rho)$, which finally implies that

$$|u(p) - u(q)| \leq |u(p') - u(q')| < \varepsilon.$$

and this conclude the proof of the theorem. □

We are now able to solve positively the problem of Plateau in the case of rectifiable curves.

Theorem 5.3.17. *Let u be a weak limit of a minimising sequence of E in $\mathscr{P}^*(\Gamma)$. Then, we have $u \in C^0(\overline{\mathbb{D}}, \mathbb{R}^n)$.*

Proof. Since u is a harmonic function, it is smooth (and even real-analytic) in the interior of $\mathbb{D}$, and continuous on the boundary $\partial\mathbb{D}$ thanks to Theorem (5.3.13). It remains to show that u remains continuous for a sequence of points of $\mathbb{D}$ that converge to the boundary $\partial\mathbb{D}$ (i.e., establish the tangential continuity).

We therefore let $\{u_k\}_{k\in\mathbb{N}}$ be a minimising sequence and assume that $u_k \hookrightarrow u$ in $W^{1,2}$ as $k \to \infty$. Now, for all $k \in \mathbb{N}$, let $v_k \in W^{1,2}(\mathbb{D})$ be the unique solution of the Dirichlet problem

$$\begin{cases} \Delta v_k = 0 & \text{in } \mathbb{D} \\ v_k = u_k & \text{on } \partial\mathbb{D} \end{cases} \tag{5.3.15}$$

Since $u_k \in W^{1,2}(\mathbb{D})$, it admits a trace in $H^{1/2}$ on $\mathbb{D}$ and the problem is solvable. Otherwise, since $u_k \in \mathscr{P}^*(\Gamma)$, $u_k \in C^0(\partial\mathbb{D})$, and we also deduce that the equation is uniquely solvable by using the Poisson kernel. Furthermore, the Dirichlet principle implies that

$$\int_{\mathbb{D}} |\nabla v_k|^2 dx \leq \int_{\mathbb{D}} |\nabla u_k|^2 dx,$$

which shows that $\{v_k\}_{k\in\mathbb{N}}$ is also a minimising sequence. Since $\{u_k\}_{k\in\mathbb{N}}$ uniformly converges to u on the boundary $\partial\mathbb{D}$, the Poisson formula (say) shows that $\{v_k\}_{k\in\mathbb{N}}$ converges in $W^{1,2}$ to v, the unique solution of

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{D} \\ v = u & \text{on } \partial\mathbb{D} \end{cases} \tag{5.3.16}$$

As u also solves this system, we deduce that $u = v$, which shows that the minimising sequence $\{v_k\}_{k\in\mathbb{N}}$ converges to u strongly in $W^{1,2}$. Now, by the maximum principle, we have

$$\|v_k - v_l\|_{\mathrm{L}^\infty(\mathbb{D})} \leq \|v_k - v_l\|_{\mathrm{L}^\infty(\partial\mathbb{D})} = \|u_k - u_l\|_{\mathrm{L}^\infty(\partial\mathbb{D})},$$

and using Theorem (5.3.13), the uniform convergence of $\{u_k\}_{k\in\mathbb{N}}$ towards u in $C^0(\partial\mathbb{D})$ show the uniform convergence of $\{v_k\}_{k\in\mathbb{N}}$ to u on $\partial\mathbb{D}$, which implies that $u \in C^0(\overline{\mathbb{D}}, \mathbb{R}^n)$. □

5.3.6 Non-triviality of the Plateau Class

The last step that we have left out in the proof is to show that the class $\mathscr{P}(\Gamma)$ is not empty. Experimentally, this is clear, but the proof of this fact will require a rather involved analysis. We will see that our previous study of Sobolev spaces and trace theory proves crucial in the analysis.

Lemma 5.3.18. *Let $u \in W^{1,2}(\mathbb{D},\mathbb{R}^d)\cap C^0(\overline{\mathbb{D}},\mathbb{R}^d)$ be a harmonic function that is weakly monotone on $\partial\mathbb{D}$ (recall Definition 5.2.2). Then, for all $0<r<1$, if $\Gamma_r = u(\partial\mathbb{D}(0,r))$ we have*

$$\mathscr{L}(\Gamma_r) = \int_0^{2\pi}\left|\frac{\partial u}{\partial\theta}(r,\theta)\right|d\theta \leq \mathscr{L}(\Gamma)$$

and

$$\lim_{r\to 1}\mathscr{L}(\Gamma_r) = \mathscr{L}(\Gamma). \tag{5.3.17}$$

Proof. Recall (exercise) that the Laplacian is given in polar coordinates in $\mathbb{R}^2$ by

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2.$$

Therefore, if u is expanded in Fourier series as

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} u_n(r)e^{in\theta},$$

the equation $\Delta u = 0$ shows that for all $n\in\mathbb{Z}$, we have

$$u_n''(r) + \frac{1}{r}u_n'(r) - \frac{n^2}{r^2}u_n(r) = 0. \tag{5.3.18}$$

Write $u_n(r) = Y_n(\log(r))$. Then, we have

$$\begin{cases} u_n'(r) = \dfrac{1}{r}Y_n' \\ u_n''(r) = \dfrac{1}{r^2}\left(Y_n''-Y_n'\right). \end{cases}$$

Therefore, the equation (5.3.18) becomes

$$0 = \frac{1}{r^2}(Y_n''-Y_n') + \frac{1}{r}\left(\frac{1}{r}Y_n'\right) - \frac{n^2}{r^2}Y_n = \frac{1}{r^2}\left(Y_n''-n^2Y_n\right),$$

and we therefore obtain the elementary equation

$$Y_n'' - n^2Y_n = 0. \tag{5.3.19}$$

The associated characteristic polynomial is given by $X^2-n^2 = (X+n)(X-n)$, which shows that for $n\neq 0$, the solutions of (5.3.19) are given by

$$Y_n(t) = \alpha_n\, e^{nt} + \beta_n e^{-nt} \qquad \alpha_n,\beta_n\in\mathbb{R}.$$

For $n=0$, the equation shows that Y_n is linear, which yields

$$Y_n(t) = \alpha_0 + \beta_0\, t \qquad \alpha_0,\beta_0\in\mathbb{R}.$$

Finally, we see that u_n admits the following expansion

$$u = \alpha_0 + \beta_0\log(r) + \sum_{n\in\mathbb{Z}^*}\left(\alpha_n r^n + \beta_n r^{-n}\right)e^{in\theta}.$$

We will use the fact that $u \in W^{1,2}(\mathbb{D})$ to show that most coefficients vanish. First, we have

$$\partial_r u = \frac{\beta_0}{r}$$

and since $\dfrac{1}{|x|} \notin L^2(\mathbb{D})$, we deduce that $\beta_0 = 0$. Then, using Parseval's identity and polar coordinates, we deduce that

$$\begin{aligned}\int_{\mathbb{D}} u^2 dx &= 2\pi\alpha_0^2 \int_0^1 r\,dr + 2\pi \int_0^1 \sum_{n\in\mathbb{Z}^*} \left|\alpha_n r^n + \beta_n\, r^{-n}\right|^2 r\,dr\,d\theta \\ &= \pi\alpha_0^2 + 2\pi \sum_{n\in\mathbb{Z}^*} \int_0^1 \left(|\alpha_n|^2 r^{2n+1} + 2\,\mathrm{Re}\,\left(\alpha_n\overline{\beta_n}\right) r + |\beta_n|^2 r^{-2n+1}\right) dr.\end{aligned}$$

Therefore, we see that for $n > 0$, we have $\beta_n = 0$, and for $n < 0$, we have $\alpha_n = 0$, which finally shows that

$$u = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n\, r^n e^{in\theta} + \sum_{n=1}^{\infty} \beta_{-n} r^n e^{-in\theta}.$$

Furthermore, u is a real-valued function, which implies that $\beta_{-n} = \overline{\alpha_n}$ and we finally get

$$u = \alpha_0 + 2\,\mathrm{Re}\,\left(\sum_{n=1}^{\infty} \alpha_n\, r^n e^{in\theta}\right) = \alpha_0 + 2\,\mathrm{Re}\,\left(\sum_{n=1}^{\infty} \alpha_n z^n\right),$$

so we recover the fact that u is the real part of a holomorphic function. For simplicity of notation, we will write $\beta_{-n} = \alpha_{-n}$ so that

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} \alpha_n r^{|n|} e^{in\theta}.$$

Now, if

$$K(r,\theta) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta},$$

we have for all $z_0 = r_0 e^{i\theta_0} \in \mathbb{D}$ the identity

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} K(r_0, \theta_0 - \theta) u(1,\theta) d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n\in\mathbb{Z}} r_0^{|n|} e^{in(\theta_0-\theta)} \sum_{m\in\mathbb{Z}} \alpha_m e^{im\theta} d\theta \\ &= \sum_{n\in\mathbb{Z}} \alpha_n r_0^{|n|} e^{in\theta_0} = u(r_0,\theta_0) = u(z_0).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}K(r,\theta) &= -1 + \sum_{n\in\mathbb{N}} z^n + \sum_{n\in\mathbb{N}} \overline{z}^n = -1 + \frac{1}{1-z} + \frac{1}{1-\overline{z}} = \frac{-|1-z|^2 + 1 - \overline{z} + 1 - z}{|1-z|^2} \\ &= \frac{-(1 - 2\,\mathrm{Re}\,(z) + |z|^2) + 2 - 2\,\mathrm{Re}\,(z)}{|1-z|^2} = \frac{1-|z|^2}{|1-z|^2}.\end{aligned}$$

Therefore, the Poisson formula is finally established: for all $z \in \mathbb{D}$, we have

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta} - z|^2} u_{\partial\mathbb{D}}(\theta) d\theta.$$

Since u is smooth, for all $0 < r < 1$, we have

$$\mathscr{L}(\Gamma_r) = \int_0^{2\pi} \left|\frac{\partial u}{\partial\theta}(r,\theta)\right| d\theta.$$

Now, as u is weakly monotone on $\partial\mathbb{D}$ and $\mathscr{L}(\Gamma)<\infty$, we deduce that Theorem 5.3.2 applies and that

$$\mathscr{L}(\Gamma)=\int_0^{2\pi}\left|\frac{\partial u}{\partial\theta}(1,\theta)\right|d\theta,$$

where this formula has to be understood in terms of Radon measures. Now, we deduce by the Poisson formula that

$$\begin{aligned}\frac{\partial u}{\partial\theta}(r,\theta)&=\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial\theta}K(r_0,\theta-\varphi)u(1,\varphi)d\varphi=-\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial\varphi}K(r_0,\theta-\varphi)u(1,\varphi)d\varphi\\&=\frac{1}{2\pi}\int_0^{2\pi}K(r,\theta-\varphi)\frac{\partial u}{\partial\varphi}(1,\varphi)d\varphi,\end{aligned}$$

where the last line is to be understood in the sense of distributions. Since

$$\frac{1}{2\pi}\int_0^{2\pi}K(r_0,\varphi)d\varphi=1,$$

we deduce by Fubini's theorem that

$$\begin{aligned}\mathscr{L}(\Gamma_r)=\int_0^{2\pi}\left|\frac{\partial u}{\partial\theta}(r,\theta)\right|d\theta&\leq\frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^{2\pi}K(r,\theta-\varphi)\left|\frac{\partial u}{\partial\varphi}(1,\varphi)\right|d\varphi\right)d\theta\\&=\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\partial u}{\partial\varphi}(1,\varphi)\right|\left(\int_0^{2\pi}K(r,\theta-\varphi)\right)d\varphi\\&=\int_0^{2\pi}\left|\frac{\partial u}{\partial\varphi}(1,\varphi)\right|d\varphi=\mathscr{L}(\Gamma).\end{aligned}$$

Therefore, we deduce that

$$\limsup_{r\to1}\mathscr{L}(\Gamma_r)\leq\mathscr{L}(\Gamma).\tag{5.3.20}$$

On the other hand, for all $\varepsilon>0$, if $\Delta=\{a_0,a_1,\cdots,a_{m-1}\}\in\mathscr{S}([0,2\pi])$ is a subdivision of $[0,2\pi]$ (recall Definition 5.3.2) such that

$$\mathscr{L}_0(u(1,\,\cdot\,),\Delta)\geq\mathscr{L}(\Gamma)-\varepsilon,$$

we have

$$\begin{aligned}\liminf_{r\to1}\mathscr{L}(\Gamma_r)\geq\liminf_{r\to1}\mathscr{L}_0(u(r,\,\cdot\,),\Delta_0)&=\liminf_{r\to1}\sum_{i=1}^{m-1}|u(r,a_i)-u(r,a_{i-1})|\\&=\sum_{i=1}^{m-1}|u(1,a_i)-u(1,a_{i-1})|\\&\geq\mathscr{L}(\Gamma)-\varepsilon,\end{aligned}$$

where we used that $u\in C^0(\overline{\mathbb{D}},\mathbb{R}^n)$. Since the result is valid for all $\varepsilon>0$, we deduce that

$$\liminf_{r\to0}\mathscr{L}(\Gamma_r)\geq\mathscr{L}(\Gamma).\tag{5.3.21}$$

Combining (5.3.20) and (5.3.21), the identity (5.3.17) is finally established.

The monotony of $r\mapsto\mathscr{L}(\Gamma_r)$ can be proven with similar methods, but we will not need it in the rest of the proof and we omit its proof. □

For all $f\in H^{1/2}(S^1)$, let $\widetilde{f}\in W^{1,2}(\mathbb{D},\mathbb{R}^n)$ be its harmonic extension, i.e., the unique function solving the equation

$$\begin{cases}\Delta\widetilde{u}=0 & \text{in } \mathbb{D}\\ \widetilde{u}=u & \text{on } \partial\mathbb{D}.\end{cases}$$

By the Douglas formula, we have

$$\int_{\mathbb{D}} |\nabla \widetilde{f}|^2 dx = \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \frac{|f(\theta)-f(\varphi)|^2}{4\sin^2\left(\frac{\theta-\varphi}{2}\right)} d\theta\, d\varphi = \|f\|_{\mathrm{H}^{1/2}(S^1)}\,.$$

If $u \in W^{1,2}(\mathbb{D}, \mathbb{R}^n)$ is a harmonic function, we define $\partial_r u \in H^{-1/2}(S^1)$ by the formula

$$\forall f \in H^{1/2}(S^1), \quad \int_0^{2\pi} \frac{\partial u}{\partial r}(1,\theta) f(\theta) d\theta = \int_{\mathbb{D}} \nabla u \cdot \nabla \widetilde{f}\, dx. \tag{5.3.22}$$

We now establish a Pohozaev-type identity for boundary values of minimisers of E in $\mathscr{P}(\Gamma)$.

Lemma 5.3.19. *Let $u \in W^{1,2}(\mathbb{D}, \mathbb{R}^n)$ be a minimum of the Dirichlet energy E in the Plateau class $\mathscr{P}(\Gamma)$. Then, we have $\frac{\partial u}{\partial r} \in L^1(S^1)$, and furthermore, we have*

$$\left|\frac{\partial u}{\partial r}(1,\theta)\right| = \left|\frac{\partial u}{\partial \theta}(1,\theta)\right| > 0 \quad \textit{for } \mathscr{L}^1 \textit{ almost every } \theta \in [0, 2\pi]$$

and

$$\frac{\partial u}{\partial r}(1,\theta) \cdot \frac{\partial u}{\partial \theta}(1,\theta) = 0 \quad \textit{for } \mathscr{L}^1 \textit{ almost every } \theta \in [0, 2\pi].$$

Proof. Since u is harmonic, we have $\Delta u = \mathrm{div}(\nabla u) = 0$ in $\mathbb{D}$, which shows by Poincaré's lemma that there exists $v \in W^{1,2}(\mathbb{D}, \mathbb{R}^n)$ such that

$$\nabla u = \nabla^{\perp} v,$$

where $\nabla^{\perp} = (\partial_{x_2}, -\partial_{x_1})$. Furthermore, v satisfies the following system of equations

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{D} \\ \left|\frac{\partial v}{\partial r}\right| = \frac{1}{r}\left|\frac{\partial u}{\partial \theta}\right| = \left|\frac{\partial u}{\partial r}\right| = \frac{1}{r}\left|\frac{\partial v}{\partial \theta}\right| & \text{in } \mathbb{D}. \end{cases} \tag{5.3.23}$$

We first establish the following inequality

$$\sup_{\theta \in [0,2\pi]} \int_0^1 \left|\frac{\partial v}{\partial r}(r,\theta)\right| dr \leq \int_0^{2\pi} \left|\frac{\partial u}{\partial \varphi}(1,\varphi)\right| d\varphi. \tag{5.3.24}$$

Since u is harmonic, the Poisson formula yields

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} K(r, \theta-\varphi) u(1,\varphi) d\varphi.$$

Introduce

$$h(r,\psi) = \int_0^{\psi} K(r,\sigma) d\sigma,$$

we get

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial h}{\partial \psi}(r,\theta-\varphi) u(1,\varphi) d\varphi \\ &= -\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial \varphi}\left(h(r,\theta-\varphi)\right) u(1,\varphi) d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} h(r,\theta-\varphi)\frac{\partial u}{\partial \varphi}(1,\varphi) d\varphi - \frac{1}{2\pi}\left(h(r,\theta-2\pi) - h(r,\theta)\right) u(1,\theta). \end{aligned}$$

Now, we have

$$h(r,\theta-2\pi)-h(r,\varphi)=\frac{1}{2\pi}\int_{\theta}^{\theta-2\pi}K(r,\sigma)d\sigma=-\int_0^{2\pi}K(r,\psi)d\psi=-2\pi,$$

which yields

$$u(r,\theta)-u(1,\theta)=\frac{1}{2\pi}\int_0^{2\pi}h(r,\theta-\varphi)\frac{\partial u}{\partial\varphi}(1,\varphi)d\varphi.$$

Therefore, we deduce that

$$\frac{\partial u}{\partial r}(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial h}{\partial r}(r,\theta-\varphi)\frac{\partial u}{\partial\varphi}(1,\varphi)d\varphi.$$

Now, we have

$$h(r,\psi)=\int_0^{\psi}\sum_{n\in\mathbb{Z}}r^{|n|}e^{in\theta}=\psi+\sum_{n\in\mathbb{Z}^*}r^{|n|}\left(\frac{e^{in\psi}-1}{i\,n}\right).$$

Therefore, we have

$$\begin{aligned}\frac{\partial h}{\partial r}(r,\psi)&=\sum_{n\in\mathbb{Z}^*}r^{|n|-1}\frac{|n|}{i\,n}\left(e^{in\psi}-1\right)=\frac{1}{i}\sum_{n\in\mathbb{Z}^*}r^{|n|-1}\mathrm{sign}(n)e^{in\psi}=\frac{1}{r\,i}\left(\sum_{n\in\mathbb{N}^*}z^n-\sum_{n\in\mathbb{N}^*}\overline{z}^n\right)\\&=\frac{1}{r\,i}\left(\sum_{n\in\mathbb{N}}z^n-\sum_{n\in\mathbb{N}}\overline{z}^n\right)=\frac{2}{r}\,\mathrm{Im}\left(\sum_{n\in\mathbb{N}}z^n\right)=\frac{2}{r}\mathrm{Im}\left(\frac{1}{1-z}\right)=\frac{2\,\mathrm{Im}\,(z)}{r|1-z|^2}.\end{aligned}$$

Therefore, we have

$$\frac{\partial h}{\partial r}(r,\psi)>0\Longleftrightarrow 0<\psi<\pi.$$

This allows us to estimate

$$\left|\frac{\partial u}{\partial r}(r,\theta)\right|\leq\frac{1}{2\pi}\int_{\theta}^{\pi+\theta}-\frac{\partial h}{\partial r}(r,\theta-\varphi)\left|\frac{\partial u}{\partial\varphi}(1,\varphi)\right|d\varphi+\frac{1}{2\pi}\int_{\pi+\theta}^{2\pi+\theta}\frac{\partial h}{\partial r}(r,\theta-\varphi)\left|\frac{\partial u}{\partial\varphi}\right|d\varphi.$$

Therefore, we get

$$\begin{aligned}\int_0^1\left|\frac{\partial u}{\partial r}(r,\theta)\right|dr\leq\frac{1}{2\pi}\int_{\theta}^{\pi+\theta}\left(h(0,\theta-\varphi)-h(1,\theta-\varphi)\right)\left|\frac{\partial u}{\partial\varphi}\right|d\varphi\\+\frac{1}{2\pi}\int_{\pi+\theta}^{2\pi+\theta}\left(h(1,\theta-\varphi)-h(0,\theta-\varphi)\right)\left|\frac{\partial u}{\partial\varphi}\right|d\varphi.\end{aligned}$$

Now, for all $\psi\in[0,2\pi]$, we have

$$|h(1,\psi)-h(0,\psi)|=\left|\int_0^{\psi}\left(K(1,\sigma)-K(0,\sigma)\right)d\sigma\right|\leq\int_0^{2\pi}K(1,\sigma)d\sigma=2\pi,$$

and the inequality (5.3.24) is established.

Now, since $v\in C^{\infty}(\mathbb{D})$ and

$$\sup_{\theta\in[0,2\pi]}\int_0^1\left|\frac{\partial v}{\partial r}(r,\theta)\right|dr<\infty,$$

the function $v(r,\theta)$ converges to a limit $v^*(\theta)$ for $\mathscr{L}^1$ almost all $\theta\in[0,2\pi]$. Now, we claim that $v^*=v(1,\,\cdot\,)$, where $v(1,\,\cdot\,)$ is the trace of v in $H^{1/2}(S^1)$.

For all subdivision $\Delta = \{a_0, \cdots, a_{m-1}\} \in \mathscr{S}([0, 2\pi])$, we have

$$\sum_{i=1}^{m-1} |v(r, a_i) - v(r, a_{i-1})| \leq \int_0^{2\pi} \left| \frac{\partial v}{\partial \theta}(r, \theta) \right| d\theta = \int_0^{2\pi} \left| \frac{\partial u}{\partial \theta}(r, \theta) \right| d\theta \leq \mathscr{L}(\Gamma).$$

Therefore, we have

$$\sup_{0<r<1} \int_0^{2\pi} \left| \frac{\partial v}{\partial \theta}(r, \theta) \right| d\theta \leq \mathscr{L}(\Gamma).$$

Since $\{v(r, \cdot\,\}_{0<r<1}$ is bounded in BV, there exists a sequence $\{r_k\}_{k\in\mathbb{N}} \subset (0,1)$ such that $r_k \underset{k\to\infty}{\longrightarrow} 0$ and $\{v(r_k, \cdot\,\}_{k\in\mathbb{N}}$ strongly converges in $L^1([0, 2\pi])$. Since $\{v(r_k, \theta)\}_{k\in\mathbb{N}}$ converges to $v^*(\theta)$ for $\mathscr{L}^1$ almost every θ, the lower-semi continuity of the BV functions, we deduce that

$$\int_0^{2\pi} \left| \frac{\partial v^*}{\partial \theta}(\theta) \right| d\theta \leq \mathscr{L}(\Gamma).$$

By unicity of the limt, we deduce that

$$\lim_{r\to 1} \int_0^{2\pi} |v(r, \theta) - v^*(\theta)| = 0.$$

Now, let $f \in C^\infty(S^1)$. The previous convergence implies in particular that

$$\lim_{r\to 1} \int_0^{2\pi} v(r, \theta) f(\theta) d\theta = \int_0^{2\pi} v^*(\theta) f(\theta) d\theta.$$

By the property of the trace of $W^{1,2}$ functions, we also have

$$\lim_{r\to 1} \int_0^{2\pi} v(r, \theta) f(\theta) d\theta = \int_0^{2\pi} v(1, \theta) \varphi(\theta) d\theta.$$

Therefore, both functions v^* and $v(1, \cdot\,)$ coincide.

Now, we establish the regularity $v(1, \cdot\,) \in W^{1,1}(S^1)$. Notice that Lemma 5.3.18 and the conformality of u show that

$$\limsup_{r\to 1} \int_{\partial\mathbb{D}(0,r)} |\nabla u| d\mathscr{H}^1 \leq 2\mathscr{L}(\Gamma) < \infty. \tag{5.3.25}$$

Introduce the holomorphic function $f = u - i\,v$. We have

$$|f'(z)|^2 = |\nabla u|^2,$$

so (5.3.25) translates to

$$\limsup_{r\to 1} \int_0^{2\pi} \left| f'(re^{i\theta}) \right| \theta < \infty.$$

This condition shows that the function f' belongs to the Hardy space $\mathcal{H}(S^1)$. Therefore, a Theorem of F. Riesz ([19]; see also [16, Theorem 3.8 p. 98] implies that there exists $g \in L^1(S^1)$ such that

$$\lim_{r\to 1} \int_0^{2\pi} \left| f'(re^{i\theta}) - g(\theta) \right| = 0.$$

Now, we have

$$\frac{\partial}{\partial \theta}(u - i\,v) = \frac{\partial}{\partial \theta} f(re^{i\theta}) = i\,r\,e^{i\theta} f'(re^{i\theta}),$$

which shows that

$$\lim_{r\to 1}\int_0^{2\pi}\left|\frac{\partial}{\partial\theta}(u-i\,v)(r,\theta)-i\,e^{i\theta}g(\theta)\right|=0$$

which implies in turn that

$$\lim_{r\to 1}\int_0^{2\pi}\left|\frac{\partial}{\partial\theta}v(r,\theta)-\mathrm{Im}\,\left(i\,e^{i\theta}g(\theta)\right)\right|=0.$$

Therefore, we have

$$\frac{\partial v}{\partial\theta}(1,\theta)=-\mathrm{Im}\,\left(i\,e^{i\theta}g(\theta)\right)\in L^1(S^1).$$

Likewise, we have

$$\frac{\partial u}{\partial r}(r,\theta)=\mathrm{Re}\,\left(e^{i\theta}g(\theta)\right)=\frac{\partial v}{\partial\theta}(1,\theta)\in L^1(S^1).$$

Recall that by definition, we have

$$\int_0^{2\pi}\frac{\partial u}{\partial r}(1,\theta)f(\theta)d\theta=\int_{\mathbb{D}}\nabla u\cdot\nabla\widetilde{f}\,dx=\int_{\mathbb{D}}\nabla^{\perp}v\cdot\nabla\widetilde{f}\,dx=\int_{\partial\mathbb{D}}\frac{\partial v}{\partial\theta}(1,\theta)f(\theta)d\theta,$$

and we finally deduce that

$$\frac{\partial u}{\partial r}(1,\theta)=\frac{\partial v}{\partial\theta}(1,\theta)\qquad\mathscr{L}^1\ \text{almost everywhere.}$$

Finally, another theorem of F. Riesz shows that either $g=0$ identically, or $|g|>0$ almost everywhere. Finally, the second equation of (5.3.23) shows that

$$\left|\frac{\partial u}{\partial\theta}(1,\theta)\right|=\frac{|g(\theta)|}{\sqrt{2}}$$

so the first claim of the lemma is entirely established. Furthermore, we have for all $0<r<1$

$$\frac{\partial u}{\partial r}\cdot\frac{\partial u}{\partial\theta}=0,$$

so the second identity follows from the afore-proved convergence. □

We can finally establish the non-emptiness of the Plateau class.

Theorem 5.3.20. *Let Γ be a rectifiable Jordan curve in $\mathbb{R}^n$. Then the Plateau class $\mathscr{P}(\Gamma)$ is non-empty.*

We start by en elementary isoperimetric inequality.

Lemma 5.3.21. *Let u be a minimiser of the Dirichlet energy in $\mathscr{P}(\Gamma)$. Then, we have*

$$E(u)\leq\frac{1}{4}\mathscr{L}^2(\Gamma).$$

Proof. Since u is harmonic, we have by an immediate integration by parts (using the definition of the integral on S^1 for $H^{1/2}$ functions)

$$\int_{\mathbb{D}}|\nabla u|^2dx=\int_0^{2\pi}\frac{\partial u}{\partial r}(1,\theta)\left(u(1,\theta)-u(1,0)\right)d\theta.$$

The previous Lemma 5.3.19 shows that

$$\frac{\partial u}{\partial r}(1,\theta)\in L^1(S^1)\qquad\text{and}\qquad\frac{\partial u}{\partial r}(1,\theta)=\frac{\partial u}{\partial\theta}(1,\theta).$$

Therefore, we get

$$\begin{aligned}\int_{\mathbb{D}}|\nabla u|^2 &\leq \int_0^{2\pi}\left|\frac{\partial u}{\partial r}(1,\theta)\right|\|u(1,\,\cdot\,)-u(0,\,\cdot\,)\|_{\mathrm{L}^\infty(S^1)}\,d\theta\\ &=\|u(1,\,\cdot\,)-u(0,\,\cdot\,)\|_{\mathrm{L}^\infty(S^1)}\int_0^{2\pi}\left|\frac{\partial u}{\partial\theta}(1,\theta)\right|d\theta. \end{aligned}\tag{5.3.26}$$

On the other hand, we have by Theorem 5.3.2

$$\int_0^{2\pi}\left|\frac{\partial u}{\partial\theta}\right|d\theta=\mathscr{L}(\Gamma).$$

Now, we trivially have

$$\|u(1,\,\cdot\,)-u(0,\,\cdot\,)\|_{\mathrm{L}^\infty(S^1)}\leq \mathrm{diam}(\Gamma)\leq\frac{1}{2}\mathscr{L}(\Gamma)\tag{5.3.27}$$

and the inequality is proven by combining (5.3.26) and (5.3.27):

$$E(u)=\frac{1}{2}\int_{\mathbb{D}}|\nabla u|^2dx\leq\frac{1}{2}\times\frac{1}{2}\mathscr{L}(\Gamma)\times\mathscr{L}(\Gamma)=\frac{1}{4}\mathscr{L}^2(\Gamma).$$

□

Remark 5.3.22. The inequality is not optimal and we actually have

$$E(u)\leq\frac{1}{4\pi}\mathscr{L}^2(\Gamma),$$

but this refinement is not needed in the proof.

We can finally complete the proof of the existence of a solution of the Plateau problem by proving Theorem 5.3.20.

Proof. (of Theorem 5.3.20) Let $\gamma:S^1\to\Gamma$ be a parametrisation such that

$$\int_0^{2\pi}|\gamma'(\theta)|\,d\theta\leq\mathscr{L}(\Gamma)<\infty.$$

Let $\varphi\in C_c^\infty(\mathbb{R}_+,\mathbb{R}_+)$ such that

$$\begin{cases}\varphi=1 \quad \text{on } [0,1]\\ \mathrm{supp}(\varphi)\subset[0,2].\end{cases}$$

Let $\varphi_\varepsilon=\varepsilon^{-1}\varphi(\varepsilon^{-1}(\,\cdot\,))$ and define its 2π-periodisation by

$$\psi_\varepsilon(t)=\sum_{k\in\mathbb{Z}}\varphi_\varepsilon(2\pi k+t).$$

Since φ_ε has compact support, the series has only finitely many non-zero terms and therefore converges uniformly. Now, define a regularisation $\gamma_\varepsilon:S^1\to\mathbb{R}^{n+2}$ of $\gamma:S^1\to\mathbb{R}^n$ by the following formula

$$\gamma_\varepsilon(\theta)=\left(\int_0^{2\pi}\psi_\varepsilon(\theta-\varphi)\gamma(\varphi)d\varphi,\varepsilon\,e^{i\theta}\right)$$

the second component is added to ensure that γ_ε is an injective immersion for all $\varepsilon>0$. Furthermore, a result from measure theory on the convergence of convolutions shows that

$$\gamma_\varepsilon\underset{\varepsilon\to 0}{\longrightarrow}\gamma \qquad \text{strongly in} \quad W^{1,1}(S^1)\cap C^0(S^1).$$

As γ_ε is smooth, its harmonic extension $u_\varepsilon = \widetilde{\gamma_\varepsilon} : \mathbb{D} \to \mathbb{R}^{n+2}$ is an element of $\mathscr{P}(\Gamma_\varepsilon)$, where $\Gamma_\varepsilon = \gamma_\varepsilon(S^1)$. Now, thanks to Lemma 5.3.21, and noticing that the strong convergence of $\{\gamma_\varepsilon\}_{\varepsilon>0}$ towards γ (as $\varepsilon \to 0$) ensures the convergence of length, we deduce that

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{D}} |\nabla u_\varepsilon|^2 \leq \limsup_{\varepsilon\to 0} \frac{1}{2}\mathscr{L}^2(\Gamma_\varepsilon) = \frac{1}{2}\mathscr{L}^2(\Gamma_\varepsilon) < \infty.$$

Furthermore, if we fix three distinct points p_1, p_2, p_3 in growing trigonometric order on S^1 choose three sequences of points $\{q_i^\varepsilon\}_{\varepsilon>0}$ $(i = 1, 2, 3)$, up to composing u_ε with a conformal map on the disk $\mathbb{D}$, we can ensure that $u_\varepsilon(p_i) = q_i^\varepsilon$. We therefore define as previously

$$\mathscr{P}^*(\Gamma_\varepsilon) = \mathscr{P}(\Gamma_\varepsilon) \cap \{u : u(p_i) = q_i \text{ for all } i = 1, 2, 3\},$$

and

$$\mathscr{P}^*_C(\Gamma_\varepsilon) = \mathscr{P}^*(\Gamma_\varepsilon) \cap \{u : E(u) \leq C\}$$

for any fixed $\mathscr{L}^2(\Gamma)/4 < C < \infty$. Now, for all $\varepsilon_0 > 0$, we introduce the following class

$$\widetilde{\mathscr{P}} = \bigcup_{0<\varepsilon<\varepsilon_0} \mathscr{P}^*_C(\Gamma_\varepsilon).$$

For $\varepsilon_0 > 0$ small enough, the proof of Theorem 5.3.13 applies *mutadis mutandis* and we deduce that $\widetilde{\mathscr{P}}$ is an equi-continuous sub-space of $C^0(\partial\mathbb{D}, \mathbb{R}^{n+2})$. Therefore, we find a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, \varepsilon_0)$ and $u \in W^{1,2}(\mathbb{D}, \mathbb{R}^n \times \{(0,0)\}) \cap C^0(\partial\mathbb{D}, \mathbb{R}^n \times \{(0,0)\})$ such that

$$\begin{cases} u_{\varepsilon_k} \underset{k\to\infty}{\rightharpoonup} u & \text{in } W^{1,2}(\mathbb{D}, \mathbb{R}^{n+2}) \\ u_{\varepsilon_k|\partial\mathbb{D}} \underset{k\to\infty}{\longrightarrow} u & \text{in } C^0(\mathbb{D}, \mathbb{R}^{n+2}) \end{cases}$$

Furthermore, the harmonicity of u_{ε_k} shows that the convergence is strong in $\mathbb{D}$ and we finally deduce that u is a harmonic and conformal map and that $u \in C^0(\overline{\mathbb{D}}, \mathbb{R}^n \times \{(0,0)\})$. As a consequence, u is an element of $\mathscr{P}(\Gamma)$, which shows the non-triviality of the Plateau class, and in fact, u is also a solution to the problem of Plateau, which concludes the chapter. □

5.4 What Next?

After solving the Plateau problem, we can ask several natural questions: what is the regularity of the solution? Are there multiple solutions? When is the solution unique? What about the existence of minimal surfaces of higher genus spanning a given contour? Those questions are generally technical (and more suitable for graduate courses) and the optimal answer is not always known, but if the boundary curve is smooth, the solution is also smooth up to the boundary (there are more precise results for curves in $C^{k,\alpha}$ due to Hildebrandt and Nitsche).

Another natural question is to try to generalise the Plateau problem in higher dimension. In this case, the so-called parametric approach does not work and we have to use new tools: either functions of bounded variations (BV) and sets of finite perimeter for codimension 1 problems, and geometric measure theory (the theory of currents or varifolds) in general (see [11] for the best introduction to the former theory).

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