MIDTERM EXAM

Probabilité Avancée

Midterm Autumn Semester 2024

November 5 2024

Length of exam : 1h 45 minutes

Permitted : no pages of personnel notes are allowed.

No calculation machines are allowed.
Please start each question on a fresh page.

Attempt all the questions

Any dishonesty will be severely sanctioned !

Write first your full name and section :

Name : ____________________ **Given name :** ____________________

Departement : ____________________

Exercice	Points
1	
2	
3	
Total points :	

1)(i) State the Monotone convergence Theorem

(ii) What is convergence in probability? Give a weak law of large numbers result. Points according to generality.

(iii) State Fatou's Lemma.

(iv) Given two probability spaces $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$, what is the product sigma field $\mathcal{F}_1 \times \mathcal{F}_2$?

(v) Is the uncountable intersection of σ fields (of subsets of a common Ω) necessarily a σ field?

(vi) For measure space $(\Omega, \mathcal{F})$, P and Q are two probabilities. If $P(A) = Q(A)\ \forall\ A \in \mathcal{A}$ where $\sigma(\mathcal{A}) = \mathcal{F}$ must $P = Q$? Justify.

(i) FOR POSITIVE RANDOM VARIABLES $X_1 \leq X_2 \cdots \leq X_n \leq X_{n+1} \ldots$

$E(X_n) \longrightarrow E(X)$ FOR $X = \lim X_n(\omega)$

(ii) $X_n \xrightarrow{P} X$ IFF $\forall \epsilon > 0$ $P(|X_n - X| > \epsilon) \xrightarrow{n\to\infty} 0$.

ONE (AMONG MANY POSSIBILITIES)

IF $(X_i)_{i\geq 1}$ ARE iid AND $\lim\limits_{x\to\infty} x P(|X_1| > x) = 0$ THEN

$\frac{S_n}{n} - \mu_n \xrightarrow{P} 0$ WHERE $S_n = \sum\limits_{i=1}^{n} X_i$ AND $\mu_n = E(X_1 I_{|X_1| \leq n})$

(iii) FOR r.v.s $(X_n)_{n\geq 0}$ POSITIVE $E(\liminf X_n) \leq \liminf E(X_n)$

(iv) $\mathcal{F}_1 \times \mathcal{F}_2 = \sigma(A \times B ; A \in \mathcal{F}_1, B \in \mathcal{F}_2)$, THE SMALLEST σ FIELD CONTAINING ALL THE SETS OF THE FORM $A \times B$ WITH $A \in \mathcal{F}_1$, $B \in \mathcal{F}_2$.

(v) YES: GIVEN $(\mathcal{F}_i)_{i\in I}$ σ-FIELDS OF SUBSETS OF Ω.

a) $\forall i\ \Omega \in \mathcal{F}_i \Rightarrow \Omega \in \bigcap_i \mathcal{F}_i$

b) $\forall A \in \bigcap_i \mathcal{F}_i$ WE HAVE $A \in \mathcal{F}_i\ \forall i$ (INTERSECTION) SO

$A^c \in \mathcal{F}_i\ \forall i$ (σ-FIELDS CLOSED UNDER COMPLEMENTARITY)

$\Rightarrow A^c \in \bigcap_i \mathcal{F}_i$

c) IF $A_1, A_2, \ldots A_n \in \bigcap \mathcal{F}_i\ \forall n \Rightarrow \forall i\ A_1, A_2 \ldots A_n \ldots \in \mathcal{F}_i$

$\Rightarrow$ (σ-FIELDS CLOSED UNDER COUNTABLE UNION) $\bigcup_n A_n \in \mathcal{F}_i\ \forall i$

$\Rightarrow \bigcup A_i \in \bigcap \mathcal{F}_i$

1)contd

So a) b), c) imply $\cap \mathcal{Y}_i$ is a σ-field

(vi) No this is not true. It is true if in addition $\mathcal{A}$ is a π system.

Counter example

$\Omega = \{1,2,3,4\}$

$\mathcal{A} = \{\{1,2\},\{2,3\}\}$ $\quad \sigma(\mathcal{A})$ contains $\{1,2\} \cap \{2,3\} = \{2\}$
$\{1,2\} \setminus \{2,3\} = \{1\}$
$\{2,3\} \setminus \{1,2\} = \{3\}$

so $\sigma(\mathcal{A}) = 2^{\Omega}$. If $P(\{1,2\}) = Q(\{1,2\}) = P(\{2,3\}) = Q(\{2,3\})$
$= 1/2$ then it could be e.g.
$P(\{1\}) = P(\{3\}) = 1/2$ or $Q(\{2\}) = 1/2 = Q(\{4\})$.

2a) Let $X_1, X_2, \dots,$ be independant identically distributed r.v.s taking values in $\mathbb{N} \cup \{-1\} \backslash \{0,1\}$, so that

$$\forall n \geq 2 \;\; P(X_1 = n) = \frac{C}{n^2 log^2(n)},$$

$$P(X_1 = -1) = 1 - \sum_n \frac{C}{n^2 log^2(n)};$$

where C is chosen so that $E(X_1) = 0$. We define $S_n = \sum_1^n X_i$. Show that

$$\frac{S_n}{n} \to 0,$$

in probability. Is it possible to extend this to a.s. convergence? Using the truncation at level $\frac{n}{log^{3/2}(n)}$ or otherwise, show that

$$\frac{S_n}{n/log(n)} \to -C,$$

in probability. Is it possible to extend this to convergence a.s.? Justify. (You may use approximations of sums via corresponding integrals without proof.)

(i) IT IS GIVEN THAT (X_i) GET SOME WEAK LAW OF LARGE NUMBERS $\frac{S_n}{n} \xrightarrow{pr} 0 = E(X_1)$

YES THIS EXTENDS TO a.s. CONVERGENCE AS WE HAVE THE STRONG LAW OF LARGE NUMBERS

(ii) CONSIDER $S_n' = \sum_{i=1}^{n} X_i'$ WHERE $X_i' = X_i I_{|X_i| \leq n/\log^{3/2} n}$ THEN (FOR n LARGE ENOUGH)

$$E(X_1') = -\sum_{m > \frac{n}{\log^{3/2} n}} m \, C/m^2 \log^2 m = -C \sum_{m > \frac{n}{\log^{3/2} n}} \frac{1}{m \log^2 m}$$

$\approx -\frac{C}{\log n}$ (WHERE $\approx$ SIGNIFIES RATIO $\to 1$ AS $n \to \infty$.)

$$E((X_1')^2) \approx C' + \sum_{2 \leq m \leq n/\log^{3/2} n} m^2 \frac{C}{m^2 \log^2 n} \sim \frac{Cn}{\log^{3/2} n} \frac{1}{\log^2 n}$$

$$= \frac{Cn}{\log^{7/2}(n)}$$

2 contd

So $E\left(\frac{S_n' - ES_n'}{n/\log n}\right)^2 \le \frac{n \cdot C\, n/\log^{7/4} n}{(n/\log n)^2} \le \frac{C}{\log^{3/4} n}$

$\longrightarrow 0$ as $n \to \infty$. So L^2 conv $\Rightarrow$ conv in pr

$$\frac{S_n' - ES_n'}{n/\log n} \xrightarrow{pr} 0 \Longrightarrow \frac{S_n'}{n/\log n} \xrightarrow{pr} -C$$

As $ES_n' \approx -\frac{Cn}{\log n}$.

Finally $P(S_n \neq S_n') \le \sum_{i=1}^{n} P(X_i \neq X_i')$

$= \quad n P\left(X_1 > \frac{n}{\log^{3/4}(n)}\right) \approx \frac{nC}{n/\log^{3/4} n \, \log^2(n/\log^{3/4} n)}$

$\approx \frac{C}{\log^{5/4}(n)} \xrightarrow{n\to\infty} 0$

So $\frac{S_n'}{n/\log n} \xrightarrow{pr} -C \Longrightarrow \frac{S_n}{n/\log n} \xrightarrow{pr} -C$

3) a) For X a positive random variable with $E(X) = 1$ on space $(\Omega, \mathcal{F}, P)$, show carefully that

$$A \to \int_A X dP$$

is a probability measure on $(\Omega, \mathcal{F})$.

b) Consider $E \subset \mathbb{R}^d$ (not necessarily measurable) and let $\mathcal{B}_E$ be the sigma field of subsets of E generated by sets of the form $O \cap E$ for open sets O in $\mathbb{R}^d$. Show that

$$\mathcal{B}_E = \{E \cap B\} \text{ for } B \text{ Borelian .}$$

If P and Q are probability measures on $(E, \mathcal{B}_E)$ so that for each open set O $P(O \cap E) = Q(O \cap E)$, is it necessarily the case that $P = Q$?

a) We must show $Q(A) = \int_A X dP$ satisfies

$Q(\Omega) = 1$, ~~for~~ $Q(A) \in [0,1]\ \forall A \in \mathcal{F}$;

$Q(\bigcup_{i=1}^{\infty} A_i) = \sum_i^{\infty} Q(A_i)$ for $A_i \in \mathcal{F}$ disjoint.

(i) $Q(\Omega) = \int_\Omega X dP = E(X) = 1$.

(ii) $\forall A$ $\quad 0 \le Q(A) = \int_A X dP \le \int_\Omega X dP = 1$

as $X \geq 0$.

(iii) Given $\{A_i\}_{i=1}^{\infty}$ disjoint

$Q(\bigcup_{i=1}^{\infty} A_i) = \int_{\cup A_i} X dP \overset{Def}{=} \int X I_{\cup A_i} dP = \int X \sum_i^{\infty} I_{A_i} dP$

as A_i are disjoint.

Let $X_n = X \sum_i^n I_{A_i}$ $\quad X' = X \sum_{i=1}^{\infty} I_{A_i}$ then

$0 \le X_n$ and $X_n \uparrow X'$ so by MCT $E(X_n) =$

$\int X \sum_i^n I_{A_i} \overset{\text{lin of } \int}{=} \sum_{i=1}^{n} \int_{A_i} X dP = \sum_i^n Q(A_i)$

increases to $\int X' dP = Q(\bigcup_i A_i)$

i.e. $\sum^{\infty} Q(A_i) = Q(\bigcup_{i=1}^{\infty} A_i)$

3contd TO SHOW $\mathcal{B}_E = \{E\cap B : B\in\mathcal{B}_d\} =: \mathcal{B}_{1E}$

WE MUST SHOW $\mathcal{B}_E \subseteq \mathcal{B}_{1E}$ AND $\mathcal{B}_E \supseteq \mathcal{B}_{1E}$

$\mathcal{B}_E \subseteq \mathcal{B}_{1E}$: CLAIM $\mathcal{B}_{1E}$ IS A σ-FIELD OF SUBSETS OF E

(1) $E\in\mathcal{B}_{1E}$ AS $E = E\cap\mathbb{R}^d$, $\mathbb{R}^d\in\mathcal{B}_d$

(2) $A\in\mathcal{B}_{1E} \Rightarrow A = B\cap E$ FOR $B\in\mathcal{B}_d$
$\Rightarrow A^c = B^c\cap E \in \mathcal{B}_{1E}$ AS $B^c\in\mathcal{B}_d$.

(3) IF $A_1, A_2, \ldots \in\mathcal{B}_{1E}$ THEN $\forall i$ $A_i = E\cap B_i$ FOR $B_i\in\mathcal{B}_d$ SO $\cup A_i = E\cap\bigcup_{i=1}^{\infty} B_i \in\mathcal{B}_{1E}$ AS $B_i\in\mathcal{B}_d \Rightarrow \cup B_i\in\mathcal{B}_d$.

SO $\mathcal{B}_{1E}$ IS A σ-FIELD. THEREFORE AS (DEF'N $\mathcal{B}_d$)
$\mathcal{B}_{1E}$ CONTAINS $O\cap E$ $\forall O$ OPEN IN $\mathbb{R}^d$
$\mathcal{B}_{1E}$ CONTAINS $\mathcal{B}_E = \sigma(O\cap E;\ O \text{ OPEN})$

$\mathcal{B}_{1E}\subseteq\mathcal{B}_E$: LET $\mathcal{H} = \{A\subseteq\mathbb{R}^d$ SO THAT $A\cap E\in\mathcal{B}_E\}$

THEN $\mathcal{H}$ CONTAINS ALL OPEN SETS, BY DEF'N OF $\mathcal{B}_E$.

CLAIM $\mathcal{H}$ IS A σ-FIELD (i) $\mathbb{R}^d\in\mathcal{H}$ AS $\mathbb{R}^d$ IS OPEN

(ii) $A\in\mathcal{H} \Rightarrow \cancel{A}\ A\cap E\in\mathcal{B}_E$
$\Rightarrow E / A\cap E = E\cap A^c \in\mathcal{B}_E$
AS $\mathcal{B}_E$ IS A σ-FIELD.
$\Rightarrow A^c\in\mathcal{H}$.

(iii) $A_1, A_2 \ldots \in\mathcal{H}$
$\Rightarrow A_1\cap E, A_2\cap E \ \ A_n\cap E .. \in\mathcal{B}_E \ \forall n$
$\Rightarrow \cup(A_i\cap E)\in\mathcal{B}_E$ AS $\mathcal{B}_E$ IS A σ-FIELD
$\Rightarrow (\cup A_i)\cap E = \cup(A_i\cap E)\in\mathcal{B}_E$
$\Rightarrow \cup A_i\in\mathcal{H}$.

SO $\mathcal{H}$ IS A σ-FIELD CONTAINING OPEN SETS $\Rightarrow \mathcal{H}$ CONTAINS $\mathcal{B}_d$
$\Rightarrow \mathcal{B}_{1E}\subseteq\mathcal{B}_E$

FINALLY

IF $Q(O \cap E) = P(O \cap E)$ $\forall$ O OPEN

THEN (AS $\Pi = \{O \cap E : O \text{ OPEN}\}$ IS A π-SYSTEM

THAT GENERATES $\mathcal{B}_E$ (BY DEFN OF $\mathcal{B}_E$)

WE HAVE BY DYNKIN THAT $Q = P$ ON $\mathcal{B}_E$

IE $Q = P$.

Π IS A π SYSTEM AS $(O_1 \cap E) \cap (O_2 \cap E)$

$= (O_1 \cap O_2) \cap E$ AND FOR O_1, O_2 OPEN

$O_1 \cap O_2$ IS OPEN.