Series 9: convergence in distribution

Exercise 1
Let $X_n, 1 \le n \le \infty$, be real random variables with respective densities $f_n, 1 \le n \le \infty$ (with respect to the Lebesgue measure). We assume that $f_n(x) \to f_\infty(x)$ for every $x \in \mathbb{R}$. Show that $X_n \to X_\infty$ in law.

Exercise 2
Let $(F_n)_{n\in\mathbb{N}}, F$ be cumulative distribution functions such that F is continuous. Show that if for every $x \in \mathbb{R}$, $F_n(x)$ converges to $F(x)$ as n tends to infinity, then $\sup_x |F_n(x) - F(x)| \to 0$.

Exercise 3
Let $X_n, 1 \le n \le \infty$ be random variables with values in $\mathbb{Z}$. Show that $X_n \to X_\infty$ in distribution if and only if $\mathbb{P}(X_n = m) \to \mathbb{P}(X_\infty = m)$ for every m. If $\mathbb{P}(X_n = m)$ is a convergent sequence for every m, does X_n converge in distribution ?

Exercise 4
Show that if $X_n \to X$ in probability, then $X_n \to X$ in distribution, and conversely, that if $X_n \to c$ in distribution where c is a constant, then $X_n \to c$ in probability.

Exercise 5
Show that if $X_n \to X$ in distribution and $Y_n \to c$ in distribution, where c is a constant, then $X_n+Y_n \to X+c$ in distribution.

Exercise 6
Let $(X_n)_{n\in\mathbb{N}}$ be independent and identically distributed random variables, and let $M_n = \max(X_1, \dots, X_n)$. What is the asymptotic behaviour of M_n as n tends to infinity, when X_1 is uniformly distributed on $[0, 1]$? What is it if X_1 follows a Cauchy distribution ? If X_1 follows an exponential distribution ?

Exercise 7
Show that

$$\rho(F, G) = \inf\{\epsilon : \ \forall x, \ F(x - \epsilon) - \epsilon \le G(x) \le F(x + \epsilon) + \epsilon\}$$

defines a metric on the space of cumulative distribution functions, and that $\rho(F_n, F) \to 0$ if and only if $F_n \to F$ at every continuity point of F.