Series 6: Borel-Cantelli lemma

Exercise 1
Let X be a real random variable, and U be a uniform random variable on $[0,1]$. Let F be the c.d.f. of X, and $F^{-1}(y) = \inf\{x : y \leqslant F(x)\}$. Show that $F^{-1}(U)$ has the same law as X. (*Remark:* this is convenient when one wants to generate random variables on a computer, since there is usually a "built-in" way to generate uniform random variables.)

Exercise 2
Show that for every sequence $(X_n)_{n\geqslant 1}$ of random variables, there exists a sequence of real numbers $c_n \to +\infty$ such that

$$\frac{X_n}{c_n} \to 0 \quad \text{a.s.}$$

Exercise 3
Let $X_1, X_2, \ldots$ be independent random variables. Show that $\sup X_n < \infty$ a.s. if and only if there exists $A > 0$ such that $\sum_n \mathbb{P}\{X_n > A\} < \infty$.

Exercise 4
Let $(X_n)_{n\geqslant 1}$ be a sequence of i.i.d. random variables such that $X_1 \sim \exp(1)$, and let $c > 0$. Compute

$$\mathbb{P}\Big[\frac{\mathrm{X_n}}{\mathrm{log(n)}} > \mathrm{c} \quad \text{for infinitely many n's} \Big].$$

What is $\limsup\limits_{n\to+\infty} X_n/\log(n)$?

Exercise 5
(i.) Show that if $\mathbb{P}\{A_n\} \to 0$ and $\sum_{n=1}^{\infty} \mathbb{P}\{A_n^c \cap A_{n+1}\} < \infty$, then $\mathbb{P}\{A_n \text{ i.o.}\} = 0$.
(ii.) Find a sequence A_n which satisfies the assumptions of (i.) but not those of the Borel-Cantelli lemma.

Exercise 6
Consider the probability space $([0,1], \mathcal{B}([0,1]), \mathbb{P} = \text{Lebesgue measure})$. For every $n \geqslant 1$, we define the random variable $X_n(\omega) = \lfloor 2^n \omega \rfloor - 2\lfloor 2^{n-1}\omega \rfloor$, where $\lfloor x \rfloor$ is the integer part of a real number x. Show that for every ω, $X_n(\omega) \in \{0,1\}$ and

$$\omega = \sum_{k=1}^{+\infty} X_k(\omega) 2^{-k}.$$

(In other words, $(X_k(\omega))_{k\in\mathbb{N}^*}$ are the coefficients of the (proper) dyadic expansion of ω.) Show that $(X_k)_{k\in\mathbb{N}^*}$ are independent and identically distributed random variables, and give their common distribution. Show that almost surely, every finite sequence of 0's and 1's appears infinitely many times in the sequence $(X_k(\omega))_{k\in\mathbb{N}^*}$.

Exercise 7
Let $\epsilon > 0$. Show that for almost every $x \in [0,1]$ (with respect to Lebesgue measure), there exists no more than a finite number of rationals p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\epsilon}}.$$

Give points for which this condition is satisfied, and others for which it is not.