Series 4: Expectation, independence

Exercise 1
Let X be a real random variable, and $1 \leqslant p \leqslant q \leqslant +\infty$. Show that $\|X\|_p \leqslant \|X\|_q$.

Exercise 2
Let X be a real random variable. Show that $\|X\|_\infty = \lim_{p\to\infty} \|X\|_p$.

Exercise 3
Let X be an integrable random variable defined on $(\Omega, \mathcal{F}, P)$, and let $(A_n)_{n\geqslant 1}$ be a sequence of measurable sets.

(a) Show that

$$\lim_{n\to\infty} \int_{|X|>n} X \, d\mathbb{P} = 0 .$$

(b) Show that if $\mathbb{P}[A_n] \to 0$, then

$$\lim_{n\to\infty} \int_{A_n} X \, d\mathbb{P} = 0 .$$

Exercise 4
Let X be a real random variable, and $p > 0$.

(a) Show that if $\mathbb{E}[\,|X|^p\,] < \infty$, then

$$\lim_{t\to+\infty} t^p \, \mathbb{P}[\,|X| \geqslant t\,] = 0 .$$

(b) Show that if $\lim_{t\to+\infty} t^p \, \mathbb{P}[\,|X| \geqslant t\,] = 0$, then

$$\mathbb{E}[\,|X|^r\,] < \infty \quad \text{for all} \quad r \in (0, p) .$$

Hint: (prove and) use the fact that if Y is a positive random variable, then $\mathbb{E}[Y] = \int_0^{+\infty} \mathbb{P}[Y \geqslant t] \, dt$.

Exercise 5
Let $X : \Omega \to \mathbb{R}$ be a real random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that X is independent of X, i.e.

$$\forall E_1, E_2 \in \mathcal{B}(\mathbb{R}), \qquad \mathbb{P}[\, X \in E_1 \ \text{ et } \ X \in E_2\,] = \mathbb{P}[\, X \in E_1\,] \, \mathbb{P}[\, X \in E_2\,].$$

Show that X is almost surely constant, i.e. there exists $a \in \mathbb{R}$ such that $\mathbb{P}[X = a] = 1$.

Exercise 6
Let $X_1, \ldots, X_n$ be random variables taking values in countable sets $S_1, \ldots, S_n$. Show that, for $X_1, \ldots, X_n$ to be independent, it is sufficient that

$$\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n) = \prod_{i=1}^{n} \mathbb{P}(X_i = x_i) \qquad \text{for all } x_i \in S_i .$$

Exercise 7

Let $X_1, \dots, X_n$ be n real random variables.

(1) Assume that for every $i \in \{1, \dots, n\}$, the distribution of X_i has density f_i, and that $X_1, \dots, X_n$ are independent random variables. Show that $(X_1, \dots, X_n)$ has a density given by

$$f(x_1, \dots, x_n) = \prod_{i=1}^{n} f_i(x_i).$$

(2) Reciprocally, assume that the distribution of $(X_1, \dots, X_n)$ has density of the form

$$f(x_1, \dots, x_n) = \prod_{i=1}^{n} g_i(x_i),$$

for some positive measurable functions g_i. Show that the random variables $X_1, \dots, X_n$ are independent and that for every i, the distribution of X_i has a density f_i which can be written as $f_i = C_i g_i$ for some constant $C_i > 0$. (*Hint: a convenient way to determine the distribution of a random variable Z is to compute* $\mathbb{E}[\varphi(Z)]$ *for nice functions* φ.)

Exercise 8

Let U be an exponential random variable of parameter 1, and V be a uniform random variable on $[0, 1]$. We assume that U and V are independent, and define

$$X = \sqrt{U} \cos(2\pi V), \quad Y = \sqrt{U} \sin(2\pi V).$$

Show that the random variables X and Y are independent.