Series 12: central limit theorem

Exercise 1
Let X and Y be independent real random variables with common characteristic function φ. Show that

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(t)|^2 \, dt = \mathbb{P}[X = Y].$$

Exercise 2
1) The aim of this question is to prove Lyapunov's theorem. Let $X_1, X_2, \ldots$ be independent random variables, and $S_n = X_1 + \ldots + X_n$. Let $\alpha_n = \{\mathrm{Var}(S_n)\}^{1/2}$. Show that if there exists $\delta > 0$ such that

$$\lim_{n\to\infty} \alpha_n^{-(2+\delta)} \sum_{m=1}^{n} \mathbb{E}(|X_m - \mathbb{E}X_m|^{2+\delta}) = 0,$$

then $(S_n - \mathbb{E}S_n)/\alpha_n \xrightarrow{(L)} \mathcal{N}(0, 1)$.
Hint. We can assume that $\mathbb{E}(X_m) = 0 \ \forall m$. Define $Y_{m,n} = X_m/\alpha_n$ and check the assumptions of the Lindeberg-Feller theorem. The following identity may be useful:

$$\mathbb{E}(Y \cdot \mathbf{1}_{(Y>\epsilon)}) \le \mathbb{E}(Y \cdot (Y/\epsilon)^{\delta} \cdot \mathbf{1}_{(Y>\epsilon)}) \le \mathbb{E}(Y \cdot (Y/\epsilon)^{\delta}).$$

2) Check the assumptions of Lyapunov's theorem for (i) $X_n \sim \mathrm{Unif}[-n, n]$; (ii) X_n with probability density $(2n)^{-1} e^{-|x|/n}$ ($x \in \mathbb{R}$).

Exercise 3
Let $X_1, \ldots, X_n$ be independent real random variables such that $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] < \infty$, and let $S_n = X_1 + \cdots + X_n$. Show Kolmogorov's maximal inequality, that is,

$$\mathbb{P}\left[\max_{1\leqslant k\leqslant n} |S_k| \geqslant x\right] \leqslant \frac{\mathbb{E}[(S_n)^2]}{x^2}.$$

Hint. Introduce the event A_k defined by

$$|S_k| \geqslant x \text{ and } \forall j < k, |S_j| < x,$$

and write

$$\mathbb{E}[S_n^2] \geqslant \sum_{k=1}^{n} \mathbb{E}[(S_k + (S_n - S_k))^2 \ \mathbb{1}_{A_k}].$$

Exercise 4
Let $X_1, X_2, \ldots$ be i.i.d. random variables, with $\mathbb{E}(X_i) = 0$ and $\mathbb{E}X_i^2 = \sigma^2 \in (0, \infty)$, and let $S_n = X_1 + \ldots + X_n$. Let N_n be a sequence of integer-valued random variables, and (a_n) a sequence of integers, with $a_n \to \infty$ and $N_n/a_n \to 1$ in probability. Show that

$$\frac{S_{N_n}}{\sigma\sqrt{a_n}} \xrightarrow{(L)} N(0, 1).$$

Hint. Define $Y_n = S_{N_n}/\sigma\sqrt{a_n}$, $Z_n = S_{a_n}/\sigma\sqrt{a_n}$ and show that $Y_n - Z_n \to 0$ in probability. Use Kolmogorov's maximal inequality.