Series 6: Borel-Cantelli lemma **Solutions**

Exercise 1
We first check that, since F is right-continuous,

$$F^{-1}(u) \leqslant y \text{ iff } u \leqslant F(y).$$

Then, we compute

$$\mathbb{P}[Y \leqslant y] = \mathbb{P}[F^{-1}(U) \leqslant y] = \mathbb{P}[U \leqslant F(y)] = F(y).$$

So the c.d.f. of Y is that of X. Since we know that a c.d.f. characterizes the law of a random variable, this finishes the proof.

Exercise 2
For each $n \geq 1$ we have

$$\lim_{t\to\infty} P[X_n > t] = 0 .$$

Then, for each $n \geqslant 1$ we can choose a constant $c_n > 0$ such that

$$P[\, X_n > \frac{c_n}{n} \,] \leq 2^{-n} .$$

In this way, we get

$$\sum_n^{+\infty} P\Big[\frac{X_n}{c_n} > 1/n \Big] < +\infty .$$

By Borel-Cantelli, we have

$$P\Big[c_n^{-1} X_n > 1/n \quad \text{i.o.} \Big] = 0 .$$

Observe that

$$\Big\{ \omega \in \Omega : c_n^{-1} X_n(\omega) > 1/n \quad \text{i.o.} \Big\}^{\mathbf{c}} \subseteq \Big\{ \omega \in \Omega : c_n^{-1} X_n(\omega) \to 0 \Big\} .$$

Then $P[c_n^{-1} X_n \to 0] = 1$.

Exercise 3
By the Borel-Cantelli lemma and independence, for all positive A we have

$$\sum_n \mathbb{P}\{X_n > A\} = \infty \Leftrightarrow \limsup X_n > A \text{ a.s..} \tag{1}$$

Suppose there exists $A^* > 0$ such that $\sum_n \mathbb{P}\{X_n > A^*\} < \infty$. Then, by (**??**), we must have $\limsup X_n \leq A^* < \infty$ almost surely, so $\sup_n X_n < \infty$ almost surely. Now suppose $\sum_n \mathbb{P}\{X_n > A\} = \infty$ for all positive A. Then, by (**??**), we must have $\mathbb{P}(\sup_n X_n > A) \geq \mathbb{P}(\limsup X_n > A) = 1$, for all positive A, and in particular

$$\mathbb{P}(\sup X_n = \infty) = \mathbb{P}\left(\bigcap_{i=1}^{\infty} \left\{ \sup_n X_n > i \right\} \right) = \lim_{i\to\infty} \mathbb{P}\left(\sup_n X_n > i \right) = 1.$$

Exercise 4
(1) A simple caculation provides $\mathbb{P}[\,\mathrm{X_n} > \mathrm{c}\log(\mathrm{n})\,] = \frac{1}{\mathrm{n^c}}$, so using the Borell-Cantelli lemma, the probability is 1 if $c \leqslant 1$, and 0 otherwise.

(2) If (u_n) is a (deterministic) sequence of real numbers, one can show that

$$u_n \geqslant c \text{ for infinitely many } n\text{'s} \Rightarrow \limsup_{n\to+\infty} u_n \geqslant c,$$

and conversely, that

$$\limsup_{n\to+\infty} u_n > c \Rightarrow u_n > c \text{ for infinitely many } n\text{'s}.$$

Using these two facts and part (1), we obtain that

$$\limsup_{n\to+\infty} \frac{X_n}{\log(n)} = 1 \text{ a.s.}$$

Exercise 5

$(i.)$ We have

$$\mathbb{P}\{A_n \text{ i.o.}\} = \lim_{m\to\infty} \mathbb{P}\{\cup_{n=m}^{\infty} A_n\} \leqslant \limsup_{m\to\infty}(\mathbb{P}\{A_m\} + \sum_{n=m+1}^{\infty} \mathbb{P}\{A_{n-1}^c \cap A_n\}) = 0.$$

$(ii.)$ We take $\Omega = [0,1], \mathcal{F} =$ borelians, $\mathbb{P} =$ Lebesegue measure. Define $A_n = [0, 1/n]$. So, $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, then we cannot use Borel-Cantelli. However, the two hypothesis of part $(i.)$ are satisfied.

Exercise 6

We have $\lfloor 2x \rfloor - 2\lfloor x \rfloor \in \{0,1\}$ (note that as a function of x, it is 1-periodic), so indeed $X_n(\omega) \in \{0,1\}$. Moreover, one can show by induction that for every n and every ω,

$$0 \leqslant \omega - \sum_{k=1}^{n} X_k(\omega) 2^{-k} < 2^{-n},$$

which ensures that

$$\omega = \sum_{k=1}^{+\infty} X_k(\omega) 2^{-k}.$$

For every sequence $i_1, \ldots, i_n \in \{0,1\}$, we see that

$$\{X_1 = i_1, \ldots, X_p = i_n\} = \left[\sum_{k=1}^{n} i_k 2^{-k}, \sum_{k=1}^{n} i_k 2^{-k} + 2^{-n}\right),$$

so

$$\mathbb{P}\left[X_1 = i_1, \ldots, X_n = i_n\right] = 2^{-n}.$$

By summing over all $i_1, \ldots, i_{n-1} \in \{0,1\}$, we thus get that

$$\mathbb{P}\left[X_n = i_n\right] = 1/2,$$

for $i_n \in \{0,1\}$. This shows that X_n is distributed as a Bernoulli random variable with parameter $1/2$. Moreover, since

$$\mathbb{P}\left[X_1 = i_1, \ldots, X_n = i_n\right] = 2^{-n} = \prod_{k=1}^{n} \mathbb{P}\left[X_k = i_k\right],$$

we have that $(X_k)_{k\in\mathbb{N}^*}$ are independent random variables. To conclude, we show that the sequence $(i_1, \ldots, i_n)$ apears infinitely often in the sequence $(X_k(\omega))_{k\in\mathbb{N}^*}$. To see this, note that for every k,

$$\mathbb{P}[X_{1+kn} = i_1, \ldots, X_{n+kn} = i_n] = 2^{-n},$$

and moreover, the events

$$(\{X_{1+kn} = i_1, \dots, X_{n+kn} = i_n\})_{k\in\mathbb{N}^*}$$

are independent (recall that "grouping" preserves independence). By the Borel-Cantelli lemma, we thus obtain that for almost every ω,

$$X_{1+kn} = i_1, \dots, X_{n+kn} = i_n \quad \text{for infinitely many } k\text{'s}.$$

This shows that every given finite sequence of 0's and 1's appears almost surely infinitely many times in the sequence $(X_k)_{k\in\mathbb{N}^*}$. Since the set of finite sequences of 0's and 1's is countable, we actually have that almost surely, every finite sequence of 0's and 1's appears infinitely many times in the sequence $(X_k)_{k\in\mathbb{N}^*}$.

Exercise 7

For the first part, apply the Borel-Cantelli lemma with the events

$$A_q = \left\{x \in [0,1] : \exists p \in \mathbb{N} \quad \left|x - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}\right\}.$$

All rational numbers satisfy the condition. Indeed, let a/b be a rational in $[0,1]$, with $a \wedge b = 1$, and assume that for $q > b$, one has

$$\left|\frac{a}{b} - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}.$$

It follows that $|aq - bp| < 1$, and thus we have $aq = bp$, and $a/b = p/q$. With $q > b$, it cannot be that p and q are relatively prime numbers.

In fact, all algebraic numbers satisfy this property, but the proof is worth a Fields medal (it is the Thue-Siegel-Roth theorem). Because algebraic numbers form a set of null measure, there are also transcendantal numbers with this property.

Both these observations were completely inaccessible to Liouville in 1844. He wanted to know whether transcendantal numbers existed or not, and proofs of transcendance for e or π were not known at that time either. He first proved a weaker version of the Thue-Siegel-Roth theorem, and then came with a number (in fact, many numbers) that could be approached *very* closely by rationals. One example is

$$\sum_{n=1}^{+\infty} 10^{-n!}.$$