Series 5: weak law of large numbers **Solutions**

Exercise 1

Since $Var(X_n) \to 0$ then, for any $\epsilon > 0$,

$$\mathrm{P}[\,|\mathrm{X_n} - \mathrm{E}[\mathrm{X_n}]\,| > \epsilon\,] \leqslant \frac{\mathrm{Var}(\mathrm{X_n})}{\epsilon^2} \to 0 \,.$$

We have just proved that $(X_n - E[X_n]) \xrightarrow{\mathrm{P}} 0$. By hypothesis, $E[X_n] \to \alpha$, and so,

$$X_n = (X_n - E[X_n]) + E[X_n] \xrightarrow{\mathrm{P}} 0 + \alpha = \alpha$$

(one can use the triangle inequality to check that for any real random variables X_n, X, Y_n, Y, if $X_n \xrightarrow{\mathrm{P}} X$ and $Y_n \xrightarrow{\mathrm{P}} Y$, then $X_n + Y_n \xrightarrow{\mathrm{P}} X + Y$).

Exercise 2

Given $\epsilon > 0$,

$$\mathbb{P}\left(\left|\frac{X_1 + \ldots + X_n}{n}\right| > \epsilon\right) \leq \frac{\mathrm{Var}(X_1 + \ldots + X_n)}{n^2\epsilon^2} \leq \frac{\sum_{i=1}^n \sum_{j=1}^n \mathbb{E}(X_i X_j)}{n^2\epsilon^2} \leq \frac{\sum_{i=1}^n \sum_{j=1}^n r(|i-j|)}{n^2\epsilon^2}$$
$$\leq \frac{nr(0) + 2n(r(1) + \ldots + r(n))}{n^2\epsilon^2} = \frac{r(0)}{n\epsilon^2} + \frac{2(r(1) + \ldots + r(n))}{n\epsilon^2} \to 0$$

as $n \to \infty$.

Exercise 3

$(i.)$ Fix $\epsilon > 0$; we have

$$\mathbb{P}\{Y_n/n > \epsilon\} = \mathbb{P}\{Y_1 > n\epsilon\} \xrightarrow{n\to\infty} 0.$$

$(ii.)$ For $\epsilon > 0$ and $n \in \mathbb{N}$, define $A_n^\epsilon = \{|Y_n|/n > \epsilon\}$. We have

$$\left\{\frac{Y_n}{n} \to 0\right\} = \bigcap_{m=1}^{\infty}\left(\left(\limsup_n A_n^{1/m}\right)^c\right),$$

so $Y_n/n \to 0$ almost surely if and only if $\mathbb{P}\left\{\limsup_n A_n^{1/m}\right\} = 0$ for each $m \geq 1$. By independence and Borel-Cantelli, this is the same as

$$\forall m, \sum_{n=1}^{\infty} \mathbb{P}\{|Y_n|/n > 1/m\} < \infty.$$

Since all variables have the same law as Y_1, this is equivalent to

$$\forall m, \sum_{n=1}^{\infty} \mathbb{P}\{|Y_1|/n > 1/m\} < \infty \Leftrightarrow$$

$$\forall m, \sum_{n=1}^{\infty} \mathbb{P}\{m|Y_1| > n\} < \infty \Leftrightarrow$$

$$\forall m, \mathbb{E}(m|Y_1|) < \infty \Leftrightarrow$$
$$\mathbb{E}(|Y_1|) < \infty.$$

Exercise 4
Let C be such that for every n, $\lambda_n \leqslant C$. A consequence of the fact that X_n follows an exponential distribution is that

$$\mathrm{Var}[X_n] = \lambda_n^2.$$

Hence,

$$\mathrm{Var}[S_n] = \sum_{k=1}^{n} \lambda_k^2 \leqslant C \sum_{k=1}^{n} \lambda_k.$$

We write $s_n = \sum_{k=1}^{n} \lambda_k$. By assumption, $s_n \to \infty$ as n tends to infinity. For every $\delta > 0$, we thus have

$$\mathbb{P}\left[|S_n - \mathbb{E}S_n| > \delta \mathbb{E}S_n\right] \leqslant \frac{\mathrm{Var}[S_n]}{(\delta \mathbb{E}S_n)^2} \leqslant \frac{C}{\delta^2 s_n}. \tag{1}$$

This shows that $S_n/\mathbb{E}[S_n]$ tends to 1 in probability, but we need to show almost sure convergence. Let

$$n_k = \inf\{n : s_n \geqslant k^2\}, \quad T_k = S_{n_k}$$

(well defined for every k since s_n tends to infinity). Since $s_n - s_{n-1} = \lambda_n \leqslant C$ (and $\mathbb{E}T_k = s_{n_k}$), we have

$$k^2 \leqslant \mathbb{E}T_k \leqslant k^2 + C. \tag{2}$$

By (1), we have

$$\mathbb{P}\left[|T_k - \mathbb{E}T_k| > \delta \mathbb{E}T_k\right] \leqslant \frac{C}{\delta^2 k^2},$$

which is summable. By the Borel-Cantelli lemma, we obtain that

$$\mathbb{P}\left[\left|\frac{T_k}{\mathbb{E}T_k} - 1\right| > \delta \text{ i.o.}\right] = 0,$$

and since $\delta > 0$ was arbitrary, this implies that

$$\frac{T_k}{\mathbb{E}T_k} \xrightarrow[k\to\infty]{\text{a.s.}} 1.$$

For n such that $n_k \leqslant n < n_{k+1}$, we have

$$\frac{T_k}{\mathbb{E}T_{k+1}} \leqslant \frac{S_n}{\mathbb{E}S_n} \leqslant \frac{T_{k+1}}{\mathbb{E}T_k}. \tag{3}$$

Note that

$$\frac{T_k}{\mathbb{E}T_{k+1}} = \frac{T_k}{\mathbb{E}T_k}\frac{\mathbb{E}T_k}{\mathbb{E}T_{k+1}}.$$

The first fraction converges to 1 almost surely, while the second one is deterministic and converges to 1 because of (2), so the product converges to 1 almost surely. A similar analysis can be performed for the right-hand side of (3), and we thus obtain the result.

Exercise 5
As $x \mapsto \frac{1}{x\log(x)}$ is decreasing on $[1, \infty)$,we have

$$\mathbb{E}(|X_i|) = \sum_{k=2}^{\infty} k \frac{C}{k^2 \log(k)} = C \sum_{k=2}^{\infty} \frac{1}{k \log(k)}$$
$$\geq C \int_2^{\infty} \frac{1}{x \log(x)} dx = C \int_{\log(2)}^{\infty} \frac{1}{u} du = \infty$$

where $u = \log(x)$. On the other hand

$$\sum_{k=n+1}^{\infty} \frac{C}{k^2 \log(k)} \leq \frac{C}{\log(n+1)} \sum_{k=n+1}^{\infty} \frac{1}{k^2} \leq \frac{C}{\log(n+1)} \int_n^{\infty} \frac{1}{x^2} dx = \frac{C}{n \log(n+1)},$$

thus, for $\lfloor x \rfloor = n$,

$$x\mathbb{P}(|X_i| > x) \leq (n+1)\mathbb{P}(|X_i| \geq (n+1)) \leq C \frac{(n+1)}{n \log(n+1)} \xrightarrow[n\to\infty]{} 0.$$

Enfin, posant $\mu_n = \mathbb{E}(X_i \mathbb{1}_{|X_i| \leq n})$, we have

$$\mu_n = \mathbb{E}(X_i \mathbb{1}_{|X_i| \leq n}) = \sum_{k=2}^{n} (-1)^k \frac{C}{k \log(k)},$$

which converges to $\mu < \infty$ by the alternating series criterion The weak law of large numbers does the rest

Exercise 6

a) We note that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, and so $\sum_{k=1}^{\infty} 2^k p_k = 1$. Thus we prove

$$\mathbb{E}(X_n) = -p_0 + \sum_{k=1}^{\infty} 2^k p_k - \sum_{k=1}^{\infty} p_k = 1 - 1 = 0.$$

b) Put $Z_k = X_k + 1$. We have $Z_k = 0$with probability p_0 and 2^k with probability p_k. We follow the approach ofexample[**?**, ex 5.7, p 44].
To apply the lwek law for triangular arrays, we must verify that

1. $\sum_{k=1}^{n} \mathbb{P}(\left|Z_{n,k}\right| > b_n) \xrightarrow[n\to\infty]{} 0$ and

2. $b_n^{-2} \sum_{k=1}^{n} \mathbb{E}(\bar{Z}_{n,k}^2) \xrightarrow[n\to\infty]{} 0$,

for a seuence $(b_n \colon n \geq 1)$ and random variables $Z_{n,k}$ and $\bar{Z}_{n,k}$ to be fixed.
To prove (1) choose $Z_{n,k} = Z_k$ for all n and note that

$$\mathbb{P}(Z_1 > 2^m) = \sum_{k=m+1}^{\infty} \frac{2^{-k}}{k(k+1)} \leq 2^{-m} \sum_{k=1}^{\infty} \frac{2^{-k}}{(m+k)(m+k+1)} \leq \frac{2^{-m}}{m^2} \sum_{k=1}^{\infty} \frac{1}{2^k} = 2\frac{2^{-m}}{m^2}.$$

It suffices then to choose $b_n = 2^{m_n}$ so that $n/(2^{m_n+1} m_n^2) \to 0$ quand $n \to \infty$ to have $n\mathbb{P}(|Z_1| > b_n) \to 0$ quand $n \to \infty$. We choose $m_n = \min\{m \colon 2^{-m} m^{-\alpha} \leq n^{-1}\}$for $0 < \alpha < 2$. We then have $b_n > 0$, $b_n \to \infty$ quand $n \to \infty$ and

$$\frac{n}{2^{m_n - 1} m_n^2} \leq \frac{2^{m_n} m_n^{\alpha}}{2^{m_n-1} m_n^2} = \frac{2}{m_n^{2-\alpha}} \xrightarrow[n\to\infty]{} 0.$$

To prove (2) put $\bar{Z}_{n,k} = Z_k \mathbb{1}_{|Z_k| \leq b_n}$ and $u_k = \frac{2^k}{k(k+1)}$.We then have

$$\mathbb{E}(\bar{Z}_{n,k}^2) = \sum_{k=0}^{m_n} 2^{2k} p_k = \sum_{k=1}^{m_n} u_k.$$

But, for $k \geq 3$,

$$\frac{u_k}{u_{k+1}} = \frac{k+2}{2k} = \frac{1}{2}\left(1 + \frac{2}{k}\right) \leq \frac{1}{2}\left(1 + \frac{2}{3}\right) = \frac{5}{6}.$$

Consequently for $3 \leq k \leq m$,

$$\frac{u_k}{u_m} = \prod_{l=k}^{m-1} \frac{u_l}{u_{l+1}} \leq \left(\frac{5}{6}\right)^{m-k}.$$

Furthermore as $u_5 = 2^5/30 \geq 1$, we deduce from the following that $u_m \geq 1$ for all $m \geq 5$. Therefore for $m \geq 5$

$$\sum_{k=1}^{m} u_k = 1 + \frac{2}{3} + \sum_{k=3}^{m} d_k \leq 2 + u_m \sum_{k=3}^{\infty} \left(\frac{5}{6}\right)^{m-k} \leq 2 + 6u_m \leq 8u_m.$$

And so,

$$\frac{\sum_{k=1}^{n} \mathbb{E}(\bar{Z}_{n,k}^2)}{b_n^2} \leq n\frac{9d_{m_n}}{b_n^2} = \frac{9n}{2^{m_n}\, m_n(m_n+1)}$$

which tends to 0 when $n \to \infty$ si $n \leq 2^{m_n} m_n^{\alpha}$ with $\alpha < 2$.
We can then calculate

$$a_n = \sum_{k=1}^{n} \mathbb{E}(\bar{Z}_{n,k}) = n \sum_{k=0}^{m_n} 2^k p_k = n \sum_{k=1}^{m_n} \frac{1}{k(k+1)} = n \sum_{k=1}^{m_n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = n\frac{m_n}{m_n+1}.$$

The weak law for triangular arrays yields

$$\frac{\sum_{k=1}^{n} Z_k - n\frac{m_n}{m_n+1}}{2^{m_n}} = \frac{S_n + n(1 - \frac{m_n}{m_n+1})}{2^{m_n}} \to 0 \text{ en probabilité.}$$

It suffcies now to show $\exists N_0$ so that

$$2^{m_n} \leq \frac{n}{\log_2(n)},\ \forall n \geq N_0 \qquad \text{and that} \qquad \frac{n(1 - \frac{m_n}{m_n+1})}{n/\log_2(n)} \to 1$$

to show $\frac{S_n - n/\log_2(n)}{n/\log_2(n)} \to 0$ in probability and thus that $\frac{S_n}{n/\log_2(n)} \to -1$ in probability.

First note that $2^{m_n-1}(m_n-1)^\alpha \le n \le 2^{m_n} m_n^\alpha$by definition (minimality) of m_n and that $\log(n) \le m_n + \alpha\log(m_n)$. We have thus $\frac{n}{\log_2(n)} \ge 2^{m_n-1}(m_n-1)^\alpha/(m_n+\alpha\log(m_n)) \ge 2^{m_n}$ if

$$(m_n-1)^\alpha \ge 2m_n + 2\alpha\log(m_n),$$

which is true for all $n \ge N_0$ if $\alpha > 1$ and N_0 is sufficiently large.Secondly

$$\frac{n(1-\frac{m_n}{m_n+1})}{n/\log_2(n)} = \frac{log_2(n)}{m_n+1} \le \frac{m_n+\alpha\log(m_n)}{m_n} \to 1.$$

We have thus shown that the weak law for pour l triangular arrays permits to prove the desired result by choosing $m_n = \min\{m\colon 2^{-m}m^\alpha \le n^{-1}\}$ with $1 < \alpha < 2$, that is with $\alpha = 3/2$ for example.

Exercise 7
See Example 2.3.2 page 69-70 in the textbook (it is Example 6.2 page 52-53 in the second edition).