Series 4: Expectation, independence **Solutions**

Exercise 1
By Hölder's inequality,

$$\mathbb{E}\left[|X|^p\right] = \mathbb{E}\left[|X|^p \times 1\right] \leqslant \mathbb{E}\left[|X|^q\right]^{p/q} \ \mathbb{E}\left[1^{p/(q-p)}\right]^{(q-p)/p} = \mathbb{E}\left[|X|^q\right]^{p/q},$$

from which we obtain

$$\mathbb{E}\left[|X|^p\right]^{1/p} \leqslant \mathbb{E}\left[|X|^q\right]^{1/q}.$$

(The inequality can also be proved using Jensen's inequality.)

Exercise 2
If $\|X\|_\infty = \infty$, then for any $A > 0$, one has $\mathbb{P}[|X| > A] > 0$, so

$$\liminf_{p\to\infty} \|X\|_p = \liminf_{p\to\infty}\left(\int |X|^p \, d\mathbb{P}\right)^{1/p} \geq \liminf_{p\to\infty}\left(\int_{|X|>A} |X|^p \, d\mathbb{P}\right)^{1/p} \geq A \liminf_{p\to\infty}(\mathbb{P}[|X| > A])^{1/p} = A,$$

and as a consequence, $\lim_{p\to\infty} \|X\|^p = \infty$.
Assume now that $\|X\|_\infty < \infty$. As $|X| \leq \|X\|_\infty$ a.s., one has

$$\limsup_{p\to\infty} \|X\|_p = \limsup_{p\to\infty}\left(\int |X|^p \, d\mathbb{P}\right)^{1/p} \leq \limsup_{p\to\infty}(\|X\|_\infty^p)^{1/p} = \|X\|_\infty.$$

Let $\epsilon > 0$. By the definition of $\|X\|_\infty$, one has $\mathbb{P}[|X| > \|X\|_\infty - \epsilon] > 0$, and thus

$$\begin{aligned}\liminf_{p\to\infty} \|X\|_p = \liminf_{p\to\infty}\left(\int |X|^p \, d\mathbb{P}\right)^{1/p} &\geq \liminf_{p\to\infty}\left(\int_{|X|>\|X\|_\infty-\epsilon} |X|^p \, d\mathbb{P}\right)^{1/p} \\ &\geq (\|X\|_\infty - \epsilon) \liminf_{p\to\infty}(\mathbb{P}[|X| > \|X_\infty - \epsilon\|])^{1/p} = \|X\|_\infty - \epsilon,\end{aligned}$$

which finishes the proof, as $\epsilon > 0$ was arbitrary.

Exercise 3

(a) Since for any $\omega \in \Omega$, we have $X(\omega) \in \mathbb{R}$, it follows that

$$\bigcap_{n\geqslant 1}\left\{\omega \in \Omega : |X(\omega)| > n\right\} = \varnothing .$$

Therefore, if we denote $Z_n := X\,\mathbf{1}_{\{|X|>n\}}$, then $Z_n(\omega) \to 0$, $\forall \omega \in \Omega$. Also, $|Z_n| \leqslant X$. We can thus use the dominated convergence theorem to get

$$\lim_{n\to\infty} \int_{|X|>n} X \, d\mathbb{P} = \lim_{n\to\infty} \mathbb{E}[Z_n] = 0 .$$

(b) Fix an arbitrary $M > 0$. We have

$$\begin{aligned}\left|\int_{A_n} X \, d\mathbb{P}\right| &\leqslant \int_{A_n} |X| \, d\mathbb{P} \\ &\leqslant \int_{A_n\cap\{|X|>M\}} |X| \, d\mathbb{P} + \int_{A_n\cap\{|X|\leqslant M\}} |X| \, d\mathbb{P} \\ &\leqslant \int_{|X|>M} |X| \, d\mathbb{P} + M\,\mathbb{P}[A_n] .\end{aligned}$$

So,

$$\limsup_{n\to\infty} \Big| \int_{A_n} X \, d\mathbb{P} \Big| \leqslant \int_{|X|>M} |X| \, d\mathbb{P} \,.$$

Letting $M \uparrow +\infty$ and using part (a), we obtain

$$\limsup_{n\to\infty} \Big| \int_{A_n} X \, d\mathbb{P} \Big| = 0 \,,$$

which in turn implies that $\int_{A_n} X \, d\mathbb{P} \to 0$.

Exercise 4

(a) The claim follows from estimate

$$t^p \, \mathbb{P}[\, |X| \geqslant t \,] = \int_{|X|\geqslant t} t^p \, d\mathbb{P} \leqslant \int_{|X|\geqslant t} |X|^p \, d\mathbb{P} \,,$$

and dominated convergence theorem.

(b) Using the hint, we can write

$$\mathbb{E}[\, |X|^r \,] = \int_0^{+\infty} \mathbb{P}[\, |X|^r \geqslant s \,] \, ds = \int_0^{+\infty} r t^{r-1} \, \mathbb{P}[\, |X| \geqslant t \,] \, dt \,.$$

We now verify that this integral is finite. By hypothesis, there exists some $N > 0$ such that

$$t^p \, \mathbb{P}[\, |X| \geqslant t \,] \leqslant 1 \,, \quad \forall t \geqslant N \,.$$

Finally,

$$\begin{aligned}
\int_0^{+\infty} r t^{r-1} \, \mathbb{P}[\, |X| \geqslant t \,] \, dt &= \int_0^{N} r t^{r-1} \, \mathbb{P}[\, |X| \geqslant t \,] \, dt + \int_N^{+\infty} r t^{r-1} \, \mathbb{P}[\, |X| \geqslant t \,] \, dt \\
&\leqslant \int_0^{N} r t^{r-1} \, dt + r \int_N^{+\infty} t^p \, \mathbb{P}[\, |X| \geqslant t \,] \Big(\frac{1}{t^{1+p-r}} \Big) \, dt \\
&\leqslant N^r + r \int_N^{+\infty} \frac{1}{t^{1+p-r}} \, dt \\
&< +\infty \,,
\end{aligned}$$

where we have used the fact that $p - r > 0$, in the last estimate.

To prove the identity in the hint, note that, writing μ for the distribution of the positive random variable Y,

$$\mathbb{E}[Y] = \int y \, d\mu(y) = \int \int_0^y dx \, d\mu(y) = \int_{x,y\geqslant 0} \mathbb{1}_{x\leqslant y} \, dx \, d\mu(y) = \int_{x\geqslant 0} \mathbb{P}[Y \geqslant x] \, dx,$$

where we used Fubini's theorem.

Exercise 5

For any $E \in \mathcal{B}(\mathbb{R})$, we have $\mathbb{P}[\, X \in E \,, X \in E \,] = \mathbb{P}[X \in E] \, \mathbb{P}[X \in E]$. So $\mathbb{P}[X \in E]$ is either equal to 0 or 1. From this, it follows that $F(x) = 0$ or 1, for each $x \in \mathbb{R}$, where $F : \mathbb{R} \to [0,1]$ is the cumulative distribution function of X.

Since $\lim_{x\to-\infty} F(x) = 0$, we know that $\{x \in \mathbb{R} : F(x) = 0\}$ is nonempty. Let us define

$$x_0 = \sup\{x \in \mathbb{R} : F(x) = 0\} \,.$$

Since $\lim_{x\to+\infty} F(x) = 1$, we also have that $x_0 < +\infty$. The function F being increasing and right-continuous, we have $F(x) = 1$ for any $x \geqslant x_0$, and $F(x) = 0$ for any $x < x_0$. Therefore, $\mathbb{P}[X = x_0] = 1$.

Exercise 6
Suppose that

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = \prod_{i=1}^{n} \mathbb{P}(X_i = x_i) \qquad \text{for } x_i \in S_i\,. \tag{1}$$

Let $E_1, E_2, \dots, E_n$ be arbitrary subsets of $S_1, \dots, S_n$. We have to prove that

$$\mathbb{P}\Big[X_1 \in E_1\,,\, X_1 \in E_1\,,\, \dots\,,\, X_n \in E_n\Big] \;=\; \mathbb{P}[X_1 \in E_1]\,\mathbb{P}[X_2 \in E_2]\,\dots\,\mathbb{P}[X_n \in E_n]\,.$$

The left-hand side in the above equality can be re-written as

$$\mathbb{P}\left[\bigcup_{x_1\in E_1,\dots,x_n\in E_n} \Big\{X_1 = x_1\,,\, X_2 = x_2\,,\, \dots\,,\, X_n = x_n\Big\}\right]$$

which in turn is equal to

$$\sum_{x_1\in F_1,\dots,x_n\in F_n} \mathbb{P}\Big[X_1 = x_1\,,\, X_2 = x_2\,,\, \dots\,,\, X_n = x_n\Big]\,.$$

Finally, the proof is completed by using (1).

Exercise 7
(1) We assume that $d\mu_{X_i}(x) = f_i(x)dx$. The independence ensures that

$$\mu_{(X_1,\dots,X_n)} = \mu_{X_1} \otimes \cdots \otimes \mu_{X_n}.$$

Fubini's theorem gives us that

$$d\left(\mu_{X_1} \otimes \cdots \otimes \mu_{X_n}\right)(x_1, \dots, x_n) = \left(\prod_{i=1}^{n} p_i(x_i)\right) dx_1 \cdots dx_n,$$

which proves the result.
(2) Since the density of $(X_1, \dots, X_n)$ is f, we can recover the density of X_i by integrating out the other variables. More precisely, X_i has a density f_i given by

$$f_i(x) = \int f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)\, dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n. \tag{2}$$

Indeed, for any positive measurable function φ, we have, by Fubini's theorem

$$\begin{aligned}\mathbb{E}[\varphi(X_i)] &= \int \varphi(x_i) f(x_1, \dots, x_n)\, dx_1 \cdots dx_n \\ &= \int \varphi(x_i)\left(\int f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)\, dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n\right) dx_i,\end{aligned}$$

which justifies the claim. We now show that there exists $C_i > 0$ such that $f_i = C_i g_i$.
Fubini's theorem and the assumption on f ensure that

$$1 = \int f(x_1, \dots, x_n)\, dx_1 \cdots dx_n = \prod_{i=1}^{n} \int g_i(x)\, dx.$$

Since $g_i \geqslant 0$, this guarantees that $K_i := \int g_i(x)\, dx$ is in $(0, +\infty)$ for every i. From (2) and the assumption of f, we obtain

$$f_i(x) = \int f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)\, dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n = \left(\prod_{j \neq i} K_j \right) g_i(x) = K_i^{-1} g_i(x),$$

so we have indeed $f_i = C_i g_i$ with $C_i = K_i^{-1}$. Finally, since

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} g_i(x_i) = \prod_{i=1}^{n} f_i(x_i),$$

it follows that $\mu_{(X_1,\ldots,X_n)} = \mu_{X_1} \otimes \cdots \otimes \mu_{X_n}$ (by Fubini's theorem), and hence the random variables $(X_1, \ldots, X_n)$ are independent.

Exercise 8

Let φ be a positive measurable function on $\mathbb{R}^2$. We have

$$\begin{aligned}
\mathbb{E}[\varphi(X, Y)] &= \int_0^{+\infty} \int_0^1 \varphi(\sqrt{u} \cos(2\pi v),\ \sqrt{u} \sin(2\pi v))\, e^{-u}\, du dv \\
&= \pi^{-1} \int_0^{+\infty} \int_0^{2\pi} \varphi(r \cos(\theta), r \sin(\theta)) r e^{-r^2}\, dr d\theta \\
&= \pi^{-1} \int_{\mathbb{R}^2} \varphi(x, y) e^{-(x^2+y^2)}\, dx dy.
\end{aligned}$$

This proves that the distribution of the pair (X, Y) has density $\pi^{-1} e^{-(x^2+y^2)}$. Since this is of product form, the previous exercise enables us to conclude that X and Y are independent.