Series 3: random variables and expectation **Solutions**

Exercise 1
First, we note that (prove it!)

$$\sigma(C) = \left\{ \bigcup_{i \in I} A_i,\ I \subseteq \{1, \ldots \kappa\} \right\}.$$

In particular, for any $i \in \{1, \ldots \kappa\}$ and any $E \in \sigma(C)$,

$$\text{if} \quad E \subseteq A_i \quad \text{then} \quad E = \varnothing \quad \text{or} \quad E = A_i\,. \tag{1}$$

Fix an arbitrary $A \in C$. Suppose that there exist two points $\omega_1, \omega_2 \in A$ such that $Y(\omega_1) \neq Y(\omega_2)$. Then,

$$\{\omega \in \Omega : Y(\omega) = Y(\omega_1)\} \cap A \ \in \ \sigma(C)$$

and

$$\{\omega \in \Omega : Y(\omega) = Y(\omega_2)\} \cap A \ \in \ \sigma(C)$$

are two nonempty disjoint subsets of A, but this contradicts (1).

Exercise 2

(a) For any $E \in \mathcal{B}(\mathbb{R})$, $(f \circ X)^{-1}(E) = X^{-1}(f^{-1}(E))$. Since $f^{-1}(E) \in \mathcal{B}(\mathbb{R})$, we have $X^{-1}(f^{-1}(E)) \in \sigma(X)$ by definition of $\sigma(X)$.

($b1$) As first step, we suppose that $Y : \Omega \to \mathbb{R}$ is simple. Then, there exists a partition $\Omega = \cup_{i=1}^n E_i$ and values $y_1, y_2, \ldots, y_n \in \mathbb{R}$ such that

$$Y(\omega) = \begin{cases} y_1 & \text{if } \omega \in E_1 \\ y_2 & \text{if } \omega \in E_2 \\ \vdots & \vdots \\ y_n & \text{if } \omega \in E_n \end{cases}$$

Furthermore, each E_i belongs to $\sigma(X)$ by the assumption on Y. By the definition of $\sigma(X)$, there exist n pairwise disjoint sets $B_i \in \mathcal{B}(\mathbb{R})$, $i = 1, 2, \ldots, n$, such that $X^{-1}(B_i) = E_i$, $\forall i = 1, \ldots, n$. Actually, the sets $(B_1, \ldots, B_n)$ form a partition of $\mathbb{R}$. Therefore, the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} y_1 & \text{if } x \in B_1 \\ y_2 & \text{if } x \in B_2 \\ \vdots & \vdots \\ y_n & \text{if } x \in B_n \end{cases}$$

is well-defined, measurable and $Y = f \circ X$.

($b2$) Now, let Y be a $\sigma(X)$-measurable function. We shall use the fact that there exists a sequence $(Y_n)_{n \geqslant 1}$ of $\sigma(X)$−measurable simple functions, such that $\lim_{n \to \infty} Y_n(\omega) = Y(\omega)$, $\forall \omega \in \Omega$. For it, it suffices define $Y_n := H_n(Y)$ where $H_n : \mathbb{R} \to \mathbb{R}$ is defined as

$$H_n(x) = \sum_{i=-n^2+1}^{n^2} \mathbf{1}_{(\frac{i-1}{n}, \frac{i}{n}]}(x)\,, \quad x \in \mathbb{R}\,,$$

for every $n \geqslant 1$.

In view of item $(b1)$, there exists a sequence $(f_n)_{n\geqslant 1}$ of measurable functions satisfying $Y_n = f_n \circ X$. Therefore, for any $\omega \in \Omega$, we have

$$f_n(\,X(\omega)\,) \to Y(\omega) \quad \text{as} \quad n \to \infty\,.$$

But this does not imply that $\lim_{n\to\infty} f_n(a)$ exists for any $a \in \mathbb{R}$. For this reason, we define

$$f(x) \;=\; \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if } x \in E \\ 0 & \text{if } x \in E^c \end{cases}$$

where $E = \{x \in \mathbb{R} : \lim_{n\to\infty} f_n(x) \text{ exists}\}$. It is easy to verify that, with this choice, f is measurable and $Y = f \circ X$.

Exercise 3

Let us write μ_X for the law of the random variable X. By definition,

$$F_X(x) = \mathbb{P}[X \leqslant x] = \mu_X\left[(-\infty, x_1] \times \cdots \times (-\infty, x_d]\right],$$

so it is clear that if $\mu_X = \mu_Y$, then $F_X = F_Y$. Reciprocally, assume that $F_X = F_Y$. Writing

$$C = \left\{(-\infty, x_1] \times \cdots \times (-\infty, x_d],\; x_1, \ldots, x_d \in \mathbb{R}^d\right\},$$

we observe that

$$\sigma(C) = \mathcal{B}(\mathbb{R}^d).$$

Since μ_X and μ_Y coincide on C and C is stable under finite intersections, we obtain that μ_X and μ_Y coincide on $\sigma(C) = \mathcal{B}(\mathbb{R}^d)$, and this finishes the proof.

Exercise 4

The "$\Leftarrow$" is clear. For the converse, a computation shows that

$$\mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = 0,$$

so $(X - \mathbb{E}[X])^2$ is a positive random variable with zero expectation. This implies that

$$\mathbb{P}\left[(X - \mathbb{E}[X])^2 = 0\right] = 1.$$

Letting $c = \mathbb{E}[X]$, we obtain the result.

Exercise 5

(a) Since $A_n \subseteq A_{n+1}$, $\forall n \geqslant 1$, then $|X|\,\mathbf{1}_{A_n}(\omega) \uparrow |X|\,\mathbf{1}_{\cup_{n\geqslant 1}A_n}(\omega)$ for every $\omega \in \Omega$. Thus, by monotone convergence theorem we obtain

$$\lim_{n\to\infty} \int_{A_n} |X|\, d\mathbb{P} \;=\; \lim_{n\to\infty} \mathbb{E}[\,|X|\,\mathbf{1}_{A_n}\,] \;=\; \mathbb{E}[\,|X|\,\mathbf{1}_{\cup_{n\geqslant 1}A_n}\,] \;=\; \int_{\cup_{n\geqslant 1}A_n} |X|\, d\mathbb{P}.$$

(b) It is enough to consider the sequence of functions

$$X_n(\omega) \;:=\; \begin{cases} X(\omega)\,, & \text{if } \;|X(\omega)| \leqslant \log(n) \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Since $|X_n| \leqslant |X|$ for all $n \geqslant 1$ and $X_n(\omega) \to X(\omega)$, $\forall \omega \in \Omega$, we have $\mathbb{E}[X_n] \to \mathbb{E}[X]$ by dominated convergence theorem (recall that X is assumed integrable).

Exercise 6

Apply Fatou's lemma on $f + f_n - |f - f_n|$.