Series 2: random variables | **Solutions**

Exercise 1
The question should have added that the two random variables X and Y take values in the *same* measurable space $(E, \mathcal{E})$... Let B be an arbitrary element of $\mathcal{E}$. We have

$$\begin{aligned} Z^{-1}(B) &= \left(Z^{-1}(B) \cap A\right) \cup \left(Z^{-1}(B) \cap A^c\right) \\ &= \left(X^{-1}(B) \cap A\right) \cup \left(Y^{-1}(B) \cap A^c\right). \end{aligned}$$

Since X is a random variable and $A \in \mathcal{F}$, we have $X^{-1}(B) \cap A \in \mathcal{F}$. Similarly, $Y^{-1}(B) \cap A^c \in \mathcal{F}$, and so $Z^{-1}(B) \in \mathcal{F}$.

Exercise 2
It suffices to observe that

$$E = \bigcup_{n \in \mathbb{N}} \left\{\omega \in \Omega : \mathbb{P}(\{\omega\}) \geqslant \frac{1}{n}\right\}$$

and that $\left|\{\omega \in \Omega : \mathbb{P}(\{\omega\}) \geqslant 1/n\}\right| \leqslant n$.
For the last assertion, let F be the cumulative distribution function of some random variable X, and let μ_X be the law of X (that is, $\mu_X(A) = \mathbb{P}[X \in A]$). Then $F(x-) \neq F(x)$ if and only if $\mu_X(\{x\}) > 0$, and the result follows apply the first observation to the probability measure μ_X.

Exercise 3
Suppose first that $X = (X_1, X_2, \ldots, X_n)$ is measurable. For any $i \in \{1, \ldots, n\}$, the projection function

$$\pi_i : \begin{cases} \mathbb{R}^n & \to \mathbb{R} \\ (x_1, \ldots, x_n) & \mapsto x_i \end{cases}$$

is continuous, and thus measurable. As a consequence, $\pi_i \circ X = \pi_i(X) = X_i$ is measurable, or in other words, X_i is a random variable.
Conversely, suppose that $X_1, X_2, \ldots, X_n$ are random variables defined on $(\Omega, \mathcal{F})$. We have seen in the previous series that

$$\mathcal{B}(\mathbb{R}^n) = \bigotimes_{i=1}^{n} \mathcal{B}(\mathbb{R}) = \sigma\Big(\{E_1 \times \cdots \times E_n,\ E_i, \in \mathcal{B}(\mathbb{R}) \text{ for all } i = 1, \ldots, n\}\Big).$$

By a result of the lecture, it is enough to prove that $X^{-1}(E_1 \times \cdots \times E_n) \in \mathcal{F}$, $\forall E_i \in \mathcal{B}(\mathbb{R})$, $i = 1, \ldots, n$. In order to do so, observe that

$$X^{-1}(E_1 \times \cdots \times E_n) = (X_1)^{-1}(E_1) \cap \cdots \cap (X_n)^{-1}(E_n).$$

Since each $(X_i)^{-1}(E_i)$ is measurable, then $X^{-1}(E_1 \times \cdots \times E_n) \in \mathcal{F}$. This concludes the proof.

Exercise 4
(1) Assume that A is measurable, and write λ for the Lebesgue measure on $\mathbb{R}$. Since

$$\mathbb{R} \subseteq \bigcup_{q \in \mathbb{Q}} (q + A),$$

it must be that $\lambda(A) > 0$, for otherwise we would conclude that $\lambda(\mathbb{R}) = 0$.

On the other hand, note that by construction, the sets $(q + A)_{q\in\mathbb{Q}}$ are disjoint. Since $A \subseteq [0, 1]$, we have

$$\bigcup_{q\in\mathbb{Q}\cap[0,1]} (q + A) \subseteq [0, 2],$$

and thus

$$\sum_{q\in\mathbb{Q}\cap[0,1]} \lambda(q + A) \leqslant 2. \tag{1}$$

Since the Lebesgue measure is invariant under translations, the measure of $\lambda(q + A)$ does not depend on q. The only possibility for (1) to hold is thus that $\lambda(A) = 0$, a contradiction with what we derived above. So A is not measurable.
(Note that we used the axiom of choice to construct A. It can be proved that the axiom of choice *has* to be used to construct non-measurable subsets of $\mathbb{R}$.)
(2) Let us consider the measurable space $(\Omega, \mathcal{F}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For every point $x \in \mathbb{R}$, the simple function $\mathbf{1}_{\{\mathbf{x}\}}$ is measurable. Let A be a subset of $\mathbb{R}$ such that $A \notin \mathcal{B}(\mathbb{R})$. Then

$$\{\mathbf{1}_{\{x\}} : x \in A\}$$

is a collection of measurable functions, but

$$\mathbf{1}_A = \sup\{\mathbf{1}_{\{x\}} : x \in A\}$$

is not measurable.

Exercise 5
We proceed by induction on the number of sets. For one subset, the claim is trivial. For two subsets, the statement is easily checked.
Now, suppose the claim is true for any k subsets. Let A_i, $i = 1, 2, \ldots, k + 1$ be arbitrary subsets of Ω. We have,

$$P\Big(\cup_{i=1}^{k+1} A_i\Big) = P\Big((\cup_{i=1}^{k} A_i) \cup A_{k+1}\Big) \tag{2}$$

$$= P\Big(\cup_{i=1}^{k} A_i\Big) + P(A_{k+1}) - P\Big((\cup_{i=1}^{k} A_i) \cap A_{k+1}\Big) \tag{3}$$

$$= P\Big(\cup_{i=1}^{k} A_i\Big) + P(A_{k+1}) - P\Big(\cup_{i=1}^{k} (A_i \cap A_{k+1})\Big) \tag{4}$$

where we have used the claim for two subsets in second equality. Finally, by using the induction hypothesis for $P(\cup_{i=1}^{k} A_i)$ and $P(\cup_{i=1}^{k} (A_i \cap A_{k+1}))$, we get the desired identity.