Series 1: probability spaces **Solutions**

Exercise 1
The distribution of X is $\mu_X : \mathcal{E} \to [0,1]$ such that for any $B \in \mathcal{E}$,

$$\mu_x(B) = \mathbb{P}[X \in B].$$

We check the defining properties of probability measures. We have $\mu_X(\emptyset) = 0$ and $\mu_X(E) = 1$. Moreover, if $(B_n)_{n\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{E}$, then

$$\begin{aligned} \mu_X\left(\bigcup_{n\in\mathbb{N}} B_n\right) &= \mathbb{P}\left[X \in \bigcup_{n\in\mathbb{N}} B_n\right] \\ &= \mathbb{P}\left[\bigcup_{n\in\mathbb{N}} \{X \in B_n\}\right]. \end{aligned}$$

Since $(B_n)_{n\in\mathbb{N}}$ are disjoint sets, so are the sets $(\{X \in B_n\})_{n\in\mathbb{N}}$ (recall that

$$\{X \in B_n\} = \{\omega \in \Omega : X(\omega) \in B_n\}),$$

and thus

$$\mu_X\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \sum_{n\in\mathbb{N}} \mu_X(B_n),$$

and this finishes the proof.

Exercise 2
Consider the σ-algebras

$$\mathcal{F}_1 = \{\{1\}, \{2,3,4\}, \emptyset, \Omega\} \quad (1)$$
$$\mathcal{F}_2 = \{\{4\}, \{1,2,3\}, \emptyset, \Omega\}, \quad (2)$$

so that

$$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\{1\}, \{4\}, \{1,2,3\}, \{2,3,4\}, \emptyset, \Omega\}.$$

$\mathcal{F}_1 \cup \mathcal{F}_2$ is not a σ-algebra because, for instance, $\{1\} \cup \{4\} \notin \mathcal{F}_1 \cup \mathcal{F}_2$.

Exercise 3
It suffices to consider the collection $C = \{\{1,2\}, \{2,3\}\}$ and

$$\begin{aligned} &\mu(\{1\}) = 1/2 && \nu(\{1\}) = 1/4 \\ &\mu(\{2\}) = 0 && \nu(\{2\}) = 1/4 \\ &\mu(\{3\}) = 1/2 && \nu(\{3\}) = 1/4 \\ &\mu(\{4\}) = 0 && \nu(\{4\}) = 1/4. \end{aligned}$$

Exercise 4
Let us write

$$C = \{(a,b) : a, b \in \mathbb{Q}\}.$$

Clearly, $C \subseteq \{$ open sets of $\mathbb{R}\}$ so $\sigma(C) \subseteq \mathcal{B}(\mathbb{R})$. On the other hand, if O is an open set of $\mathbb{R}$, we have

$$O = \bigcup_{\substack{I \subseteq O \\ I \in C}} I.$$

Since C is countable, in particular the union above is countable, and thus $O \in \sigma(C)$. We have shown that any open set belongs to $\sigma(C)$. Since $\sigma(C)$ is a σ-algebra by definition, this implies that $\mathcal{B}(\mathbb{R}) \subseteq \sigma(C)$. In conclusion, we have shown $\mathcal{B}(\mathbb{R}) = \sigma(C)$, with C countable.
For the second part, recall that

$$\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) = \sigma\left(\{A \times B,\ A, B \in \mathcal{B}(\mathbb{R})\}\right).$$

It is easy to check that for any $A \in \mathcal{B}(\mathbb{R})$, one has $A \times \mathbb{R} \in \mathcal{B}(\mathbb{R}^2)$ (check that the set of all such A's contains the open sets and is a σ-algebra). Similarly, for any $B \in \mathcal{B}(\mathbb{R})$, we have $\mathbb{R} \times B \in \mathcal{B}(\mathbb{R}^2)$. For any $A, B \in \mathcal{B}(\mathbb{R})$, we thus have

$$A \times B = (A \times \mathbb{R}) \cap (\mathbb{R} \times B) \in \mathcal{B}(\mathbb{R}^2).$$

This justifies that $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}(\mathbb{R}^2)$.
Conversely, let us define the countable collection

$$C = \{(a, b) \times (c, d),\ a, b, c, d \in \mathbb{Q}\}.$$

Let O be an open set of $\mathbb{R}^2$. We can write it as

$$O = \bigcup_{\substack{I \subseteq O \\ I \in C}} I,$$

and the union is over a countable index set. Moreover, it is clear that any $I \in C$ belongs to $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$, so we have proved that O belongs to $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$. As a consequence, $\mathcal{B}(\mathbb{R}^2) \subseteq \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$, and this concludes the proof.

Exercise 5
We first show that $\mathcal{F} = \{A$ such that A or A^c is countable $\}$ is a σ-algebra.
It is clear that $\Omega \in \mathcal{F}$ and that if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$. Now suppose that $A_n \in \mathcal{F}, n \geq 1$. If every A_n is countable, $\cup_{n \geq 1} A_n$ will also be countable. On the other hand, if A_m^c is countable for at least one m, then $(\cup_{n \geq 1} A_n)^c = \cap_{n \geq 1} A_n^c \subseteq A_m^c$ will be countable.
We now show that P is σ-additive. Indeed, suppose that the A_n are two by two disjoint. If every A_n is countable, $\cup_{n \geq 1} A_n$ will also be countable. So $P(\cup_{n \geq 1} A_n) = 0 = \sum_{n \geq 1} P(A_n)$. On the other hand, if A_m^c is not countbale for at least one m, then, since the A_n are disjoint, $A_n \subseteq A_m^c$ for each $n \neq m$. Thus, $P(A_n) = 0$ for each $n \neq m$ and we then have $P(\cup_{n \geq 1} A_n) = \sum_{n \geq 1} P(A_n) = P(A_m) = 1$.

Exercise 6
We define

$$E_k = \{i \in \mathbf{N} : 2^{2k-1} < i \leqslant 2^{2k}\} \quad \text{and} \quad F_k = \{i \in \mathbf{N} : 2^{2k} < i \leqslant 2^{2k+1}\} \quad \text{for } k \geqslant 1\ .$$

Observe that $\#E_k = 2^{2k-1}$ and $\#F_k = 2^{2k}$ for every $k \geqslant 1$.

1. For (a) consider the set $A = \cup_{k \geqslant 1} E_k$. For $n = 2^{2m}$ we have

$$\frac{\#(A \cap \{1, 2, \ldots, 2^{2m}\})}{2^{2m}} = \frac{\sum_{k=1}^{m} \#E_k}{2^{2m}} = \left(\frac{2}{3}\right)\left(\frac{2^{2m} - 1}{2^{2m}}\right)$$

and so

$$\lim_{m \to \infty} \frac{\#(A \cap \{1, 2, \ldots, 2^{2m}\})}{2^{2m}} = \frac{2}{3}\ .$$

A similar computation gives

$$\lim_{m\to\infty} \frac{\#(A\cap\{1,2,\ldots,2^{2m+1}\})}{2^{2m+1}} = \frac{1}{3}\,.$$

From this, we conclude that the whole sequence does not converge.

2. $\mathcal{A}$ is not an algebra and, consequently, it is not a σ-algebra. Indeed, let P and I be the set of even and odd numbers, respectively. We can consider the subsets

$$B \;=\; \Big(\cup_{k\geqslant 1}\,(E_k\cap P)\Big)\bigcup\Big(\cup_{k\geqslant 1}\,(F_k\cap I)\Big)\,.$$

It is not difficult to see that

$$\lim_{n\to+\infty} \frac{\#(P\cap\{1,2,\ldots,n\})}{n} = \frac{1}{2} \quad \text{and} \quad \lim_{n\to+\infty} \frac{\#(B\cap\{1,2,\ldots,n\})}{n} = \frac{1}{2}$$

and then B and P belong to $\mathcal{A}$. On the other hand, $B\cap P = \Big(\cup_{k\geqslant 1} E_k\Big)\cap P = A\cap P$ and so $B\cap P\notin\mathcal{A}$ by a similar argument to the previous item.