Exercise Series 14 28/5/2024

Exercise 1. (Extension of the Feynman-Kac Formula)

Let $q, f, \sigma : \mathbb{R} \longrightarrow \mathbb{R}$ be bounded Borel functions. Assume that the equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}(t,x) + \mu(x)\frac{\partial u}{\partial x}(t,x) + q(x)u(t,x), \quad t \in \mathbb{R}_+,\, x \in \mathbb{R},$$

with the initial condition $u(0,x) = f(x)$, $x \in \mathbb{R}$, admits a unique bounded solution, such that $\frac{\partial u}{\partial x}$ is bounded.

Assume that $\mu : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz function and therefore there exists a constant $C \in \mathbb{R}_+$ such that for all $x \in \mathbb{R}$,

$$|\mu(x)| \leqslant C(1 + |x|).$$

Show that

$$u(t,x) = \mathbb{E}\left(f\left(X_t^{(x)}\right)\exp\left(\int_0^t q\left(X_s^{(x)}\right)\,ds\right)\right),$$

where $(X_t^{(x)})$ is the unique solution of the stochastic differential equation

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t, \quad X_0 = x.$$

Exercise 2. (Properties of Brownian motion)

Let B_t be a d-dimensional Brownian motion. Show that:

(i) In $d = 1$ Brownian motion is *recurrent*, i.e. for any $x \in \mathbb{R}$ there is a (random) increasing sequence $t_n \to \infty$ such that $B_{t_n} = x$.

(ii) In $d = 2$ Brownian motion is *neighbourhood recurrent*, i.e. for every $x \in \mathbb{R}^2$ and $\epsilon > 0$, the ball $B(x, \epsilon)$ is visited infinitely often.

(iii) In $d \geqslant 3$ Brownian motion is *transient*, i.e. it converges to infinity almost surely. *Hint: consider* $A_n = \{|B_t| > \sqrt{n} \quad \forall t \geqslant T_n\}$ *where* T_n *is the hitting time of* $B(0,n)^c$.

Hint: Use the link of Brownian motion and a solution to a Dirichlet problem. In particular, note that:

$$u(x) = \phi(|x|) = \begin{cases} c_1|x| + c_2 & \text{if } d = 1 \\ c_1 \log|x| + c_2 & \text{if } d = 2 \\ c_1|x|^{2-d} + c_2 & \text{if } d \geqslant 3 \end{cases}$$

solves $\Delta u = 0$ *on* $\mathbb{R}^d \backslash \{0\}$.