
Exercise sheet 9

16/4/2025

Exercise 1.

Let $(B_t, t \in \mathbb{R}_+)$ be a standard Brownian motion. We define $M_t := B_t^2 - t$ and $N_t := \exp(B_t - \frac{t}{2})$. These two processes are martingales (see Exercise 2 Series 3). Provide the Doob decomposition of M_t^2 and N_t^2 , then deduce $\langle M \rangle_t$ and $\langle N \rangle_t$.

Exercise 2. (Ornstein-Uhlenbeck Process)

Let $(B_t, t \in \mathbb{R}_+)$ be a standard Brownian motion, $a \in \mathbb{R}^*$ fixed and $X_t := \int_0^t e^{-a(t-s)} dB_s$, $t \in \mathbb{R}_+$.

- (a) Calculate $\mathbb{E}(X_t)$ and $\text{Cov}(X_s, X_t)$.
- (b) Is the process (X_t) a martingale?
- (c) Prove that (X_t) satisfies the following equation:

$$X_t = -a \int_0^t X_s ds + B_t.$$

Exercise 3.

Let $(B_t, t \geq 0)$ be a standard Brownian motion. Show that the following processes are martingales

- (a) $X_t = e^{\frac{t}{2}} \sin(B_t)$,
- (b) $X_t = (B_t + t)e^{-B_t - \frac{t}{2}}$.

Exercise 4.

Let $(B_t, t \geq 0)$ be a standard Brownian motion.

- (a) Let $\mu \in \mathbb{R}$. Consider the Itô process $X_t = B_t + \mu t$. What is the distribution of the random variable $Z = \int_0^1 s dX_s$?
- (b) Show that $Z = B_1 + \frac{\mu}{2} - \int_0^1 B_s ds$.