Exercise 8 9/4/2024

Exercise 1.

Let $\mathcal{M}_{T,\mathrm{loc}}$ be the set of continuous local martingales $M = (M_t)_{t\in[0,T]}$. For all $M \in \mathcal{M}_{T,\mathrm{loc}}$ there exists a unique (up to indistinguisibility) process $\langle M\rangle = (\langle M\rangle_t)_{t\in[0,T]}$ with continuous, non-decreasing paths such that $\langle M\rangle_0 = 0$ and $(M_t^2 - \langle M\rangle_t)_{t\geqslant 0} \in \mathcal{M}_{T,\mathrm{loc}}$. For $M, N \in \mathcal{M}_{T,\mathrm{loc}}$ we set

$$\langle M, N\rangle := \frac{1}{4}\langle M + N\rangle - \frac{1}{4}\langle M - N\rangle.$$

One shows that $\langle M, N\rangle$ is the unique process on $[0, T]$ with continuous paths of bounded variation such that $\langle M, N\rangle_0 = 0$ and $MN - \langle M, N\rangle$ is a local martingale (see Exercise 3, Sheet 6).

Prove that for all $M, N \in \mathcal{M}_{T,\mathrm{loc}}$:

(a) $\langle M, M\rangle = \langle M\rangle = \langle -M\rangle$.

(b) $\langle M, N\rangle = \langle N, M\rangle$.

(c) $\langle M - M_0, N\rangle = \langle M, N - N_0\rangle = \langle M - M_0, N - N_0\rangle = \langle M, N\rangle$.

(d) $\langle M^\tau, N^\tau\rangle = \langle M, N\rangle^\tau$ for every stopping time τ.

(e) $(N, M) \mapsto \langle M, N\rangle$ is a bilinear map.

(f) Integration by parts for local martingales:

$$M_tN_t = M_0N_0 + \int_0^t M_s\,\mathrm{d}N_s + \int_0^t N_s dM_s + \langle M, N\rangle_t.$$

(g) If $H \in \mathcal{H}_{T,\mathrm{loc}}(M)$ and $G \in \mathcal{H}_{T,\mathrm{loc}}(N)$, then

$$\left\langle \int H_s\,\mathrm{d}M_s, \int G_s\,\mathrm{d}N_s \right\rangle = \int H_sG_s\,\mathrm{d}\langle M, N\rangle_s\,.$$

Exercise 2.

Let $M \in \mathcal{M}_{T,\mathrm{loc}}$ and $f : \mathbb{R} \to \mathbb{R}$ be a polynomial. Using the integration by parts and substitution formulas prove the following identity

$$f(M_t) = f(M_0) + \int_0^t f'(M_s)\,\mathrm{d}M_s + \frac{1}{2}\int_0^t f''(M_s)\,\mathrm{d}\langle M\rangle_s$$

known as the Ito formula (actually, the identity is true for all $f \in C^2(\mathbb{R})$).

Exercise 3.

Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion.

(a) Using Ito formula to show that $(B_t^2 - t,\, t \in \mathbb{R}_+)$ is a martingale (cf. Exercise 3, Sheet 2).

(b) Using Ito formula to show that

$$X_t = B_t^4 - 6 \int_0^t B_s^2 \, ds$$

is a martingale.