Exercise sheet 7 2/4/2024

Exercise 1.

Let $(B_t)_{t\geqslant 0}$ be a standard Brownian motion.

(a) For all $n \geqslant 1$, define $\tau_n = \inf\left\{t > 0 : \int_0^t e^{2B_s^4}\,ds \geqslant n\right\}$. Show that for all $t > 0$, $\mathbb{P}\{\tau_n \leqslant t\} > 0$.

(b) Explain how to determine a random variable $\int_0^t e^{B_s^4}\,dB_s$ from simple predictable process.

Exercise 2.

(a) Show that if $\mathcal{F}$ is a sigma-algebra and if $(X_n)_{n\in\mathbb{N}}$ is a sequence of random variables such that $X_n \underset{n\to\infty}{\longrightarrow} X$ a.s. and $|X_n| \leqslant Y \in L^1$, then

$$\mathbb{E}(X_n \mid \mathcal{F}) \underset{n\to\infty}{\longrightarrow} \mathbb{E}(X \mid \mathcal{F}), \text{ in } L^1 \text{ and a.s.}$$

(b) Let $(X_t,\, t \in \mathbb{R}_+)$ be a continuous local martingale and $T > 0$ such that $E\left(\sup_{0\leqslant s\leqslant T} |X_s|\right) < +\infty$. Show that $(X_t,\, t \in [0,T])$ is a martingale.

Exercise 3.

Let $M = (M_t,\, t \in \mathbb{R}_+)$ be a bounded continuous martingale. Define

$$S_t^n = \sum_{i=1}^n \left(M_{t_i^{(n)}} - M_{t_{i-1}^{(n)}}\right)^2$$

where $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = t$ and $\lim_{n\to\infty} \max_{1\leqslant i\leqslant n} \left(t_i^{(n)} - t_{i-1}^{(n)}\right) = 0$. Show that

$$\lim_{n\to\infty} S_t^n = M_t^2 - M_0^2 - 2\int_0^t M_s dM_s = \langle M\rangle_t, \quad \text{in } L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

Exercise 4.

Let $H \in \mathcal{H}_T$. Then $\langle \int_0^t H_s dB_s\rangle_t = \int_0^t H_s^2 ds$.