Exercise 6 26/3/2025

Exercise 1.

Let $(H_t,\, t \in [0,T])$ be a predictable process such that $\mathbb{E}(H_t^2) < +\infty$ for all $t \in [0,T]$, and $(s,t) \mapsto \mathbb{E}(H_s H_t)$ is a continuous function on $[0,T]^2$. Show that

$$\int_0^T H_s \, dB_s = \lim_{n\to\infty} \sum_{i=1}^n H_{t_{i-1}^{(n)}} (B_{t_i^{(n)}} - B_{t_{i-1}^{(n)}}) \qquad \text{in } L^2,$$

where $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = T$ and $\lim\limits_{n\to\infty} \max\limits_{1 \leqslant i \leqslant n} \left(t_i^{(n)} - t_{i-1}^{(n)} \right) = 0$.

Exercise 2.

Let $(B_t,\, t \geqslant 0)$ be a standard Brownian motion. Define

$$Z = \int_0^1 s \, dB_s.$$

(a) Is this integral well-defined?

(b) What's the value of $\mathbb{E}(Z)$ and $\mathrm{Var}(Z)$?

(c) What is the law of Z?

Exercise 3. (Polarization identity and definition of quadratic covariation)

1) Let X, Y be two square integrable random variables. Show that

$$\mathrm{Cov}(X,Y) = \frac{1}{4} \left(\mathrm{Var}(X+Y) - \mathrm{Var}(X-Y) \right).$$

2) Let (X_t), (Y_t) be two continuous and square integrable martingales with respect to the same filtration $(\mathcal{F}_t)$. We define the *quadratic variation* of (X_t) and (Y_t) as:

$$\langle X, Y \rangle_t := \frac{1}{4} \left(\langle X+Y \rangle_t - \langle X-Y \rangle_t \right), \quad t \in \mathbb{R}_+.$$

Show that the process $(X_t Y_t - \langle X, Y \rangle_t, t \in \mathbb{R}_+)$ is a martingale with respect to $(\mathcal{F}_t)$.