**Exercise 5** 19/3/2024

**Exercise 1.**

Let (X, Y) be a Gaussian vector on $\mathbb{R}^2$ such that $\mathbb{E}(X) = \mathbb{E}(Y) = 0$ and $\mathbb{E}(Y^2) > 0$. Show that $\mathbb{E}(X|Y) = \dfrac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)} Y$.

**Exercise 2.**

1) Let $(M_t, t \in \mathbb{R}_+)$ be a square integrable continuous martingale. Prove that $(M_t)$ is an orthogonal increments process, that is, it satisfies the following property:

$$\mathbb{E}((M_t - M_s)\, M_s) = 0, \quad \forall t > s \geqslant 0.$$

Deduce that $\mathrm{Cov}(M_t, M_s)$ is a function of $t \wedge s$.

2) Let $(X_t, t \in \mathbb{R}_+)$ be a centered Gaussian process satisfying the Markov property and

$$\mathbb{E}((X_t - X_s)\, X_s) = 0, \quad \forall t > s \geqslant 0.$$

Show that $(X_t)$ is a martingale respect to its natural filtration $(\mathcal{F}_t^X)$.

**Exercise 3.**

1) Let $(\mathcal{F}_t)_t$ be a filtration. Let $s < t$ and $A \in \mathcal{F}_s$. Prove that the r.v. $\tau$ defined as

$$\tau(\omega) = \begin{cases} s & \text{if } \omega \in A \\ t & \text{if } \omega \in A^c \end{cases}$$

is a stopping time.
2) Prove that an integrable continuous process $(X_t)_t$ adapted to $(\mathcal{F}_t)_t$ is a martingale if and only if, for every bounded $(\mathcal{F}_t)_t$-stopping time $\tau$, $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$.

**Exercise 4. (Stochastic integral of a simple predictable process with respect to a martingale)**

Let $(M_t,\, t \in \mathbb{R}_+)$ be a continuous martingale such that $\mathbb{E}(M_t^2) < +\infty$, for all $t \in \mathbb{R}_+$. Given a simple predictable process $(H_t,\, t \in \mathbb{R}_+)$ such that

$$H_t = \sum_{i=1}^{n} X_i\, \mathbf{1}_{]t_{i-1}, t_i]}(t),$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ and $X_i$ is $\mathcal{F}_{t_{i-1}}$-mesurable and bounded for all $i = 1, \ldots, n$. We denote

$$(H \cdot M)_T = \sum_{i=1}^{n} X_i \left(M_{t_i} - M_{t_{i-1}}\right).$$

(a) Show that $H \mapsto (H \cdot M)_T$ is linear.

(b) Show that $\mathbb{E}\left((H \cdot M)_T\right) = 0$.

(c) Let $(\langle M \rangle_t,\, t \in \mathbb{R}_+)$ be the quadratic variation of $(M_t)$. Show that

$$\mathbb{E}\left((H \cdot M)_T^2\right) = \mathbb{E}\left(\int_0^T H_s^2 \, d\langle M \rangle_s\right).$$