Exercise sheet 4 12/3/2024

Exercise 1.

Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion. For $a,\, b > 0$, define

$$T_{-a,b} = \inf\{t \geqslant 0 :\, B_t > b \text{ or } B_t < -a\}.$$

Here we assume that $T_{-a,b}$ is a stopping time.

(a) Show that $B_{T_{-a,b}} \in \{-a, b\}$, a.s.

(b) Show that $\mathbb{E}(B_{T_{-a,b}}) = 0$ and deduce that $\mathbb{P}\{B_{T_{-a,b}} = b\} = \frac{a}{a+b}$.

(c) Show that $\mathbb{E}(B^2_{T_{-a,b}}) = \mathbb{E}(T_{-a,b})$ and deduce that$\mathbb{E}(T_{-a,b}) = ab$.

Exercise 2.

Let $p > 0$. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function. Define

$$V_p(t) = \limsup_{n\to+\infty} \sum_{j=1}^{2^n} \left| f(jt2^{-n}) - f((j-1)t2^{-n}) \right|^p .$$

Let $q > p$. Show that if $V_p(t) < +\infty$, then $V_q(t) = 0$. We note that the contra-posed statement is : If$V_q(t) > 0$, then $V_p(t) = +\infty$.

Exercise 3. (Integral of a step function with respect to a function with bounded variation)

(a) Let $g :\, \mathbb{R}_+ \to \mathbb{R}$ such that $g = g_1 - g_2$, where g_1 and g_2 are increasing functions. Show that by a direct calculation, g is a function with (locally) bounded variation.

(b) Let g be a continuous function with bounded variation and let $0 \leqslant t_1 < t_2 \leqslant t$. Show that
$$\int_0^t \mathbf{1}_{[t_1,t_2]}(s)\, dg(s) = g(t_2) - g(t_1).$$

(c) Let g be a continuous function with bounded variation. Let $0 = t_0 < t_1 < \cdots < t_n = t$ and f be a step function defined by
$$f(s) = \sum_{i=1}^{n} a_i \mathbf{1}_{[t_{i-1},t_i]}(s), \text{ for } a_i \in \mathbb{R},\ i = 1, \ldots, n.$$

Deduce from point (b) that

$$\int_0^t f(s)\,dg(s) = \sum_{i=1}^{n} a_i(g(t_i) - g(t_{i-1})).$$