Exercise sheet 3 5/3/2025

Exercice 1. (Properties of stopping time)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ be a filtration. Show following properties.

(a) If T is a stopping time, then for all $t \geqslant 0$, events $\{T < t\}$, $\{T \geqslant t\}$, $\{T > t\}$ belong to $\mathcal{F}_t$.

(b) If S and T are two stopping times, then $S \vee T$ and $S \wedge T$ are stopping times.

(c) If $(T_n,\, n \in \mathbb{N})$ is a sequence of stopping times, then $\sup_{n\in\mathbb{N}} T_n$ is a stopping time.

(d) If a filtration is right continuous (i.e.,for all $t \geqslant 0$, $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$), and if $(T_n,\, n \in \mathbb{N})$ is a sequence of stopping times, then $\inf_{n\in\mathbb{N}} T_n$ is a stopping time.

Exercice 2.

Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion and $(\mathcal{F}_t^B,\, t \in \mathbb{R}_+)$ its natural filtration. Show that the processes $(B_t^2 - t,\, t \in \mathbb{R}_+)$ and $(\exp\{B_t - \frac{t}{2}\},\, t \in \mathbb{R}_+)$ are martingales with respect to $(\mathcal{F}_t^B)$.

Exercice 3. (Doob Martingale)

Let X be an integrable random variable and $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ is a filtration. Define $M_t = \mathbb{E}(X|\mathcal{F}_t)$.

(a) Show that $(M_t,\, t \in \mathbb{R}_+)$ is a martingale with respect to $(\mathcal{F}_t)$.

(b) Let $s \in \mathbb{R}_+$. What can we say about the processes (M_t) if X is $\mathcal{F}_s$-mesurable?

(c) (bonus) Show that the family M_t is uniformly integrable i.e.

$$\lim_{\lambda\to\infty} \sup_{t\geqslant 0} \mathbb{E}\left[|M_t| \mathbb{1}_{|M_t|>\lambda}\right] = 0$$

Exercice 4.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t,\ t \in [0,1])$ and let H^2 be the set of continuous martingales $M = (M_t,\ t \in [0,1])$ equipped with norm $||M||^2 = \mathbb{E}(M_1^2)$. Show that$H^2$ is a Hilbert space.