Exercise sheet 2 26/2/2025

Exercise 1.

Assume that we already know how to construct a process $(B_t^{(1)},\ t \in [0,1])$ satisfying conditions (a) and (b) of standard Brownian motion such that $t \mapsto B_t^{(1)}(\omega)$ is continuous on $[0,1]$. Construct a process $(B_t,\ t \geqslant 0)$ which is a standard Brownian motion such that $t \mapsto B_t(\omega)$ is a continuous function on $\mathbb{R}^+$.

Exercise 2.

Let X, Y, Z be random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$, $\mathcal{H}$ are sub-algebra of $\mathcal{F}$. Please prove the following results:

(a) If $\alpha, \beta \in \mathbb{R}$, then $\mathbb{E}[\alpha X + \beta Y|\mathcal{G}] = \alpha\mathbb{E}[X|\mathcal{G}] + \beta\mathbb{E}[Y|\mathcal{G}]$ a.s.

(b) If $X \leqslant Y$ a.s., $\mathbb{E}[X|\mathcal{G}] \leqslant \mathbb{E}[Y|\mathcal{G}]$ a.s.

(c) If $X_n \geqslant 0$ and $X_n \uparrow X$ a.s., then $\mathbb{E}[X_n|\mathcal{G}] \uparrow \mathbb{E}[X|\mathcal{G}]$ a.s.

Exercise 3. **(Following Exercice 2)**

(d) If $\phi : \mathbb{R} \to \mathbb{R}$ is convex and $\mathbb{E}[|\phi(X)|] < \infty$, show that $\phi(\mathbb{E}[X|\mathcal{G}]) \leqslant \mathbb{E}[\phi(X)|\mathcal{G}]$ a.s.

(e) $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[X]$.

(f) If $\mathcal{G} \subset \mathcal{H}$, $\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}] = \mathbb{E}[X|\mathcal{G}]$ a.s.

(g) If Z is $\mathcal{G}$-mesurable and $\mathbb{E}[|XZ|] < \infty$, then $\mathbb{E}[XZ|\mathcal{G}] = Z\,\mathbb{E}[X|\mathcal{G}]$ a.s. Specifically, $\mathbb{E}[Z|\mathcal{G}] = Z$ a.s.

(h) If X is independent with $\mathcal{G}$, then $\mathbb{E}[X|\mathcal{G}] = E[X]$ a.s.

Exercise 4.

Let $(B_t, t \geqslant 0)$ be a standard Brownian motion. Show that

$$\lim_{t\to\infty} \frac{B_t}{t} = 0 \quad \text{a.s.}$$