Exercise sheet 1 19/2/2025

Exercise 1.

Let $(B_t)_{t\in[0,1]}$ be a Brownian motion. Define the Brownian bridge $(X_t)_{t\in[0,1]}$ by

$$X_t = B_t - tB_1.$$

Show that the probability density function of X_t at fixed $t \in (0,1)$ is given by

$$f_{X_t}(x) = \frac{1}{\sqrt{2\pi t(1-t)}} \exp\left(-\frac{x^2}{2t(1-t)}\right).$$

Are the increments of $(X_t)_{t\in[0,1]}$ independent, that is, are X_s and $X_t - X_s$ independent for $0 \leqslant s < t \leqslant 1$.

Exercise 2.

Let $(X_n)_{n\geqslant 0}$ be an i.i.d sequence of $\mathcal{N}(0,1)$ random variables. For $l, p \in \mathbb{N}$ with $l < p$, define

$$T_{l,p} = \sup_{t\in[0,\pi]} \left| \sum_{n=l}^{p-1} X_n \frac{\sin(nt)}{n} \right|.$$

Show that

$$\mathbb{E}\left((T_{l,p})^2\right) \leqslant \frac{p-l}{l^2} + 2\frac{(p-l)^{\frac{3}{2}}}{l^2}.$$

Exercise 3.

Let $(B_t)_{t\in[0,1]}$ be a Brownian motion with initial point x and $p(t,x,\cdot)$ be its density. Show that

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}.$$

Exercise 4.

Let $(B_t,\ t \geqslant 0)$ be a Brownian motion. Fix $s > 0$ and define

$$X_t = B_{s\wedge t} - (B_t - B_{s\wedge t}) = \begin{cases} B_t, & t \leq s \\ 2B_s - B_t, & t > s. \end{cases}$$

Draw a picture of the processes $(B_t,\ t \geqslant 0)$ and $(X_t,\ t \geqslant 0)$ and show that $(X_t,\ t \geqslant 0)$ is again a Brownian motion.