
Exercise Series 13 21/5/2024

Exercise 1. Dambis-Dubins-Schwartz Theorem

Let M be a $(\Omega, \mathcal{F}_t, \mathbb{P})$-continuous local martingale, vanishing at 0 and such that $\langle M, M\rangle_\infty = \infty$. Set

$$T_t := \inf\{s \colon \langle M, M\rangle_s > t\}.$$

Show that $B_t := M_{T_t}$ is a $(\mathcal{F}_{T_t})$-Brownian motion, and $M_t = B_{\langle M,M\rangle_t}$.

Hint: You can use the fact that intervals of constancy for M and $\langle M, M\rangle$ are the same, that is, for almost all ω's, $M_t(\omega) = M_a(\omega)$ for $a \leqslant t \leqslant b$ if and only if $\langle M, M\rangle_a(\omega) = \langle M, M\rangle_b(\omega)$.

Exercise 2.

Consider the SDE

$$dX_t = \sigma(X_t)dB_t.$$

Assuming that $M > \sigma(x) > 1$ for all x, show that this SDE has a weak solution.

Hint: Use the previous exercise. You might find considering the process $M_t = \int_0^t \frac{1}{\sigma(W_s)} dW_s$ for a BM W_t useful.

Now combine this with an exercise from the previous sheet to construct a weak solution to

$$dX_t = f(X_t)dt + \sigma(X_t)dB_t.$$

Assuming f, σ are bounded.
Hint: You will need to use a more general version of Girsanov theorem which was stated with a proof outline in exercise class. It is also the statement that appears for example on Wikipedia if you google Girsanov theorem.

(**)For some extra fun, you can show that this solution is unique in Law.

Exercise 3.

Let $T > 0$. Let (B_t) be a standard Brownian motion. For each of the random variables X below, find a process $H \in \mathcal{H}_T(B)$ such that

$$X = \mathbb{E}(X) + \int_0^T H_s \, dB_s. \qquad (*)$$

(a) $X = B_T$

(b) $X = \int_0^T B_s \, ds$

(c) $X = B_T^2$

(d) $X = B_T^3$.