1 Sheet 2

1.1 Excercise 1

Prove (at least for matrix groups) that the exponential

$$\mathfrak{g} \to G, \qquad X \to \exp(X)$$

is invariant under the adjoint action, i.e.

$$\exp\big(gXg^{-1}\big) = g\exp(X)g^{-1} \tag{1}$$

for any $X \in \mathfrak{g}l_n$ and $g \in GL_n$.

Solution. Let $X \in \mathfrak{gl}_n$ and $g \in GL_n$. Then

$$\exp(gXg^{-1}) = \sum_{n=0}^{\infty} \frac{(gXg^{-1})^n}{n!} = \sum_{n=0}^{\infty} g\frac{X^n}{n!}g^{-1} = g\exp(X)g^{-1}.$$

1.2 Excercise 2

Then prove for any Lie group G that the abstractly defined exponential $\exp(X) = \gamma_X(1)$ satisfies

$$\exp(\mathrm{Ad}_g(X)) = \mathrm{Ad}_g(\exp(X)) \tag{2}$$

for any $g \in G$, $X \in \mathfrak{g}$.

Solution. Let $X \in \mathfrak{g}$, and set $X' = \mathrm{Ad}_{g,*}(X) \in \mathfrak{g}$, for some $g \in G$. Let $\gamma_X(t)$ and $\gamma_{X'}(t)$ be the associated one parameter groups given by Excercice 4 in Sheet 1. We prove that the curve $\mathrm{Ad}_g(\gamma_X(t))$ coincides with $\gamma_{X'}(t)$, so that (2) follows from evaluating in $t = 1$. By uniqueness, this amounts to verifying that both curves solve the same differential equation

$$\begin{cases} \dfrac{d}{dt}\gamma(t) = X'_{\gamma(t)} \\ \gamma(0) = e. \end{cases} \tag{3}$$

It is enough to compute the derivative of $\mathrm{Ad}_g(\gamma_X(t))$. This is

$$\frac{d}{dt}\mathrm{Ad}_g(\gamma_X(t)) = \frac{d}{ds}\Big|_{s=0}\mathrm{Ad}_g(\gamma_X(t+s)) = \mathrm{Ad}_{g,*}(X_{\gamma_X(t)}), \tag{4}$$

where the first equality holds because γ_X is a one parameter subgroup. The claim then follows from the equality

$$\mathrm{Ad}_{g,*}(X_h) = X'_{ghg^{-1}}$$

between the left-invariant vector fields associated to X and X'.

1.3 Excercise 3

Let $F : G \to G'$ be a Lie group homomorphism, and $f = F_* : \mathfrak{g} \to \mathfrak{g}'$ the induced linear map of tangent spaces. Show that we have

$$\boxed{\exp(f(X)) = F(\exp(X))} \tag{5}$$

for all $X \in \mathfrak{g}$. Conclude that f preserves the Lie bracket we defined last week

$$[X, X'] = \frac{\partial}{\partial t}\frac{\partial}{\partial t'} \exp(tX)\exp(t'X')\exp(tX)^{-1}\exp(t'X')^{-1}\Big|_{t=t'=0} \tag{6}$$

in the sense that

$$\boxed{f([X, X']) = [f(X), f(X')]} \tag{7}$$

for all $X, X' \in \mathfrak{g}$.

Solution. The proof of

$$\exp(f(X)) = F(\exp(\mathrm{X})) \tag{8}$$

is analogous to that of Excercise 2. Namely, one has to prove that

$$F(\gamma_X(t)) = \gamma_{f(X)}(t). \tag{9}$$

Finally, since F and f are differentiable morphisms, we get

$$F([X, X']) = \frac{d^2}{dtdt'}\Big|_{t,t'=0} F(\exp(\mathrm{tX}))\, F(\exp(\mathrm{t'X'}))\, F(\exp(\mathrm{tX}))^{-1}\, F(\exp(\mathrm{t'X'}))^{-1}$$
$$= \frac{d^2}{dtdt'}\Big|_{t,t'=0} \exp(tf(X))\exp(t'f(X'))\exp(tf(X))^{-1}\exp(t'f(X'))^{-1} = [f(X), f(X')].$$

1.4 Excercise 4

Consider the adjoint representaton $G \to GL(\mathfrak{g})$ and take its derivative

$$\mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \qquad \mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g} \tag{10}$$

for all $X \in \mathfrak{g}$. Then show that the Lie bracket (6) satisfies

$$[X, X'] = \mathrm{ad}_X(X') \tag{11}$$

for all $X, X' \in \mathfrak{g}$.

Solution. We use the statement of Excercise 3 in the case at hand, that is

$$\exp(\mathrm{ad}_X) = \mathrm{Ad}_{\exp(X),*} \in GL(\mathfrak{g}). \tag{12}$$

We identify $GL(\mathfrak{g}) \cong GL_n$ and $End(\mathfrak{g}) = \mathfrak{gl}_n$ by choosing any basis of $\mathfrak{g}$ as a vector space. Then for t small enough we have

$$\exp(\mathrm{ad_X}) = 1 + t\,\mathrm{ad}_X + \frac{t^2}{2}\,\mathrm{ad}_X^2 + \ldots \tag{13}$$

Evaluating (12) in any $X' \in \mathfrak{g}$ and taking the derivative in both sides yields

$$\mathrm{ad}_X(X') = \frac{d}{dt}\Big|_{t=0}\left(\mathrm{Ad}_{\exp(tX).*}(X')\right) \tag{14}$$
$$= \frac{d}{dt}\Big|_{t=0}\frac{d}{dt'}\Big|_{t'=0} \exp(tX)\exp(t'X')\exp(tX)^{-1} = [X, X'], \tag{15}$$

where we used Excercise 5 in sheet 1.

1.5 Excercise 5

The following famous result of Baker-Campbell-Hausdorff shows how to reconstruct the multiplication in a Lie group G from the Lie bracket of Lie(G), at least in a neighborhood of the identity element.

If G is a Lie group, and $X, Y \in Lie(G)$ are close enough to 0, then

$$\exp(X)\exp(Y) = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{\substack{a_1,\dots,a_n \geq 0 \\ b_1,\dots,b_n \geq 0}}^{a_i+b_i>0,\forall i} \frac{[X,\dots,[X,[Y,\dots,[Y,\dots,[X,\dots,[X,[Y,\dots,Y]\dots]}{(a_1+\dots+a_n+b_1+\dots+b_n)a_1!\dots a_n!b_1!\dots b_n!}\right) \tag{16}$$

where the inner sum involves the iterated Lie bracket of a_1 copies of X, followed by b_1 copies of Y, ..., followed by a_n copies of X, followed by b_n copies of Y.

Reverse engineer formula (16) as follows: suppose you're working in $G = GL_n$ and you want to find Z such that $\exp(X)\exp(Y) = \exp(Z)$, and Z is given by linear combinations of commutators of X and Y. Find the parts of Z which are linear, then quadratic, then cubic ... in X, Y (do so explicitly up to whatever order you can).

Solution. If $g \in GL_n$ is sufficiently close to the identity matrix 1, the series

$$\log(g) = (g-1) - \frac{(g-1)^2}{2} + \frac{(g-1)^3}{3} - \dots \tag{17}$$

converges in $\mathfrak{gl}_n$. This tells us that the exponential map is invertible near the identity[1]. Then, after rescaling,

$$Z = \log(\exp(X)\exp(Y)) \tag{18}$$

will be well defined as an element of $\mathfrak{gl}_n$. We then compute

$$\exp(X)\exp(Y) = 1 + (X+Y) + \left(\frac{X^2}{2} + XY + \frac{Y^2}{2}\right) + \left(\frac{X^3}{3!} + \frac{XY^2}{2} + \frac{YX^2}{2} + \frac{Y^3}{3!}\right) + \dots$$

and we substitute in (17). Then, our matrix Z is

$$(X+Y) + \left(\frac{X^2}{2} + XY + \frac{Y^2}{2}\right) + \left(\frac{X^3}{3!} + \frac{XY^2}{2} + \frac{YX^2}{2} + \frac{Y^3}{3!}\right) + \dots \tag{19}$$

$$-\frac{(X+Y)^2}{2} - \left((X+Y)\left(\frac{X^2}{2} + XY + \frac{Y^2}{2}\right)\right) - \dots \tag{20}$$

$$+\frac{(X+Y)^3}{3} + \dots \tag{21}$$

$$\dots \tag{22}$$

where we find the terms of the same degree in the same coloumn. Summing along the coloumns,

$$Z = (X+Y) + \frac{1}{2}[X,Y] \pm \frac{1}{12}[X,[X,Y]] \pm \frac{1}{12}[Y,[Y,X]] \pm \dots \tag{23}$$

1.6 Excercise 6

If the Baker-Campbell-Hausdorff formula was too much fun for you, then consider the following formula due to Campbell If G is a Lie group, and $X, Y \in$ Lie(G), then

$$\mathrm{Ad}_{\exp(X)}(Y) = Y + [X,Y] + \frac{[X,[X,Y]]}{2!} + \frac{[X,[X,[X,Y]]]}{3!} + \dots$$

[1] Actually, this yields a diffeomorphism between a neighborhood of $0 \in \mathfrak{g}$ and a neighborhood of $e \in G$. See [1].

and prove it for $G = GL_n$.

Solution. We need to compute

$$\exp(\mathrm{X})\, Y \exp(-X) = \left(Y + XY + \frac{X^2Y}{2} + \frac{X^3Y}{3!} + \dots\right)\left(1 - X + \frac{X^2}{2} - \frac{X^3}{3!} + \dots\right).$$

As before we organize the terms of the same degree in coloumns

$$\begin{aligned} Y - YX + \frac{YX^2}{2} - \frac{YX^3}{3!} + \dots\dots \\ XY - XYX + \frac{XYX^2}{2} - + \dots \\ \frac{X^2Y}{2} - \frac{X^2YX}{2} + \dots \\ \frac{X^3Y}{3!} + \dots \end{aligned}$$

Summing along the coloumns we get the claim.

Alternativly, one can use the equation

$$\mathrm{Ad}_{\exp(X)}(Y) = \exp^{\mathrm{ad}_X}(Y).$$

References

[1] Brian C. Hall. Lie groups, lie algebras, and representations: An elementary introduction. 2004.