Math 429 - Exercise Sheet 12

1. Check the Serre relation

$$\mathrm{ad}_{E_i}^{1-c_{ij}}(E_j) = 0$$

for the classical Lie algebras $\mathfrak{sl}_n, \mathfrak{o}_n, \mathfrak{sp}_{2n}$ (where E_i denote generators of the simple root spaces).

Solution. The root system associated to the Lie algebra $\mathfrak{sl}_n$ is A_n. We choose the simple roots $\{\alpha_i = e_i - e_{i+1}\}_{i=1,\dots,n-1}$ with the usual notation. With this convention, the generator of the simple root space associated to the root α_i is the elementary matrix $E_{i,i+1}$ (again with the usual notations). Moreover, the associated Cartan matrix is

$$\begin{bmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & \dots & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ & & & \ddots & & -1 \\ 0 & \dots & \dots & \dots & -1 & 2 \end{bmatrix}.$$

Then for $j = i \pm 1$ the Serre relation in the statement reduces to

$$\mathrm{ad}_{E_i}^{1-c_{ij}}(E_j) = [E_{i,i+1}, [E_{i,i+1}, E_{j,j+1}]] = 0, \tag{1}$$

and for other values of j

$$\mathrm{ad}_{E_i}^{1-c_{ij}}(E_j) = [E_{i,i+1}, E_{j,j+1}] = 0. \tag{2}$$

The root system associated to the Lie algebra $\mathfrak{so}_{2n}$ is D_n. We make the same choice of simple roots as in Sheets 10 and 11. We worked out in Sheet 9 that the generators of the root spaces associated to the simple roots $\beta_k = e_k - e_{k+1},\ k = 1, \dots, n-1$ are the matrices $C^3_{k,k+1}$. Recall that $C^3_{k,k+1}$ is defined as the 2×2 block matrix having

$$C^3 = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$$

in the $(k, k+1)$th block and $-(C^3)^T$ in the $(k+1, k)$th block. Moreover, the generator of the root space associated to the simple root $\beta_n = e_{n-1} + e_n$ is the matrix $C^1_{(n-1,n)}$ from Sheet 9, having

$$C^1 = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$$

in the $(n-1, n)$th block and $-(C^3)^T$ in the $(n, n-1)$th block. Finally recall the Cartan matrix from Sheet 11. Knowing these data, we can verify Serre relation in the statement. Observe that for $1 \le i, j \le n-1$ these relations are the same as those of $\mathfrak{sl}_n$ in (1) and (2), modulo substituting

$$E_{i,i+1} \rightsquigarrow C^3_{i,i+1} \quad \text{and} \quad E_{j,j+1} \rightsquigarrow C^3_{j,j+1}. \tag{3}$$

These can be verified by computation. Concerning the last root, we have to verify that

$$[C^3_{k,k+1}, C^1_{n-1,n}] = 0 \text{ for all } k \neq n-2,$$

and

$$[C^3_{n-2,n-1}, [C^3_{n-2,n-1}, C^1_{n-1,n}]] = 0.$$

We compute the last equation explicitely

$$\left[\begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & C^3 & 0 \\ & & -(C^3)^T & 0 & 0 \\ & & 0 & 0 & 0 \end{bmatrix}, \left[\begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & C^3 & 0 \\ & & -(C^3)^T & 0 & 0 \\ & & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & 0 & 0 \\ & & 0 & 0 & C^1 \\ & & 0 & -(C^1)^T & 0 \end{bmatrix}\right]\right] =$$

$$\left[\begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & C^3 & 0 \\ & & -(C^3)^T & 0 & 0 \\ & & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & 0 & 0 \\ & & 0 & 0 & C^3C^1 \\ & & 0 & (C^1)^T(C^3)^T & 0 \end{bmatrix}\right] =$$

$$\begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & 0 & 0 \\ & & 0 & 0 & -(C^3)^T C^3 C^1 \\ & & 0 & -(C^1)^T(C^3)^T C^3 & 0 \end{bmatrix} = 0.$$

The procedure for the other simple Lie algebras $\mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$ is the same, knowing the generators for every root space from Sheet 9 and the Cartan matrices from Sheet 11.

2. Check the well-definedness of the action in Proposition 30.

Solution. Let

$$\phi\colon \widetilde{\mathfrak{g}}_C \longrightarrow \mathfrak{gl}(TV)$$

be the representation from Proposition 30. We check that ϕ is a homomorphism of Lie algebras. Clearly $\phi(H_i)$ and $\phi(H_j)$ commute because they are diagonal operators. Now we compute

$$[\phi(E_i), \phi(F_j)](v_{j_1} \otimes \cdots \otimes v_{j_n}) =$$

$$\phi(E_i) v_j \otimes v_{j_1} \otimes \cdots \otimes v_{j_n} - \phi(F_j) \sum_{1 \leq s \leq n \text{ s.t. } j_s = i} (c_{ij_{s+1}} + \cdots + c_{ij_n})(v_{j_1} \otimes \cdots \otimes v_{j_{s-1}} \otimes v_{j_{s+1}} \otimes \cdots \otimes v_{j_n}) =$$

$$\delta_{ij}(c_{ij_1} + \cdots + c_{ij_n})(v_{j_1} \otimes \cdots \otimes v_{j_n}) +$$

$$+ v_j \otimes \sum_{1 \leq s \leq n \text{ s.t. } j_s = i} (c_{ij_{s+1}} + \cdots + c_{ij_n})(v_{j_1} \otimes \cdots \otimes v_{j_{s-1}} \otimes v_{j_{s+1}} \otimes \cdots \otimes v_{j_n}) -$$

$$- \phi(F_j) \sum_{1 \leq s \leq n \text{ s.t. } j_s = i} (c_{ij_{s+1}} + \cdots + c_{ij_n})(v_{j_1} \otimes \cdots \otimes v_{j_{s-1}} \otimes v_{j_{s+1}} \otimes \cdots \otimes v_{j_n}) =$$

$$\delta_{ij} \phi(H_i)(v_{j_1} \otimes \cdots \otimes v_{j_n}).$$

Moreover,

$$[\phi(H_j), \phi(E_i)](v_{j_1} \otimes \cdots \otimes v_{j_n}) =$$

$$\phi(H_j) \sum_{1 \leq s \leq n \text{ s.t. } j_s = i} (c_{ij_{s+1}} + \cdots + c_{ij_n})(v_{j_1} \otimes \cdots \otimes v_{j_{s-1}} \otimes v_{j_{s+1}} \otimes \cdots \otimes v_{j_n}) -$$

$$\phi(E_j)(c_{j,j_1} + \cdots + c_{j,j_n})(v_{j_1} \otimes \cdots \otimes v_{j_n}) = -$$

$$\sum_{1 \leq s \leq n \text{ s.t. } j_s = i} (c_{ij_{s+1}} + \cdots + c_{ij_n})(c_{j,j_1} + \cdots + c_{j,j_{s-1}} + c_{j,j_{s+1}} + \cdots + c_{j,j_n})(v_{j_1} \otimes \cdots \otimes v_{j_{s-1}} \otimes v_{j_{s+1}} \otimes \cdots \otimes v_{j_n})$$

$$+ \sum_{1 \leq s \leq n \text{ s.t. } j_s = i} (c_{ij_{s+1}} + \cdots + c_{ij_n})(c_{j,j_1} + \cdots + c_{j,j_n})(v_{j_1} \otimes \cdots \otimes v_{j_{s-1}} \otimes v_{j_{s+1}} \otimes \cdots \otimes v_{j_n}) =$$

$$c_{j,i}\phi(E_i)(v_{j_1} \otimes \cdots \otimes v_{j_n}).$$

Finally, the relation

$$[\phi(H_j), \phi(F_i)] = -c_{j,i}\phi(F_i)$$

is the easiest to check.

3. Prove the formula

$$[F_k, \mathrm{ad}_{E_i}^{1-c_{ij}}(E_j)] = 0 \tag{4}$$

in the Lie algebra $\widetilde{\mathfrak{g}}_C$, for any $i \neq j$ and k. In other words, show that the Lie brackets above are 0 by using only antisymmetry, the Jacobi identity, and relations (154)-(157). *Hint: show that*

$$ad_{E_i}^{1-c_{ij}}(E_j) = \sum_{t=0}^{1-c_{ij}} (-1)^t \binom{1-c_{ij}}{t} E_i^t E_j E_i^{1-c_{ij}-t} \tag{5}$$

in $U(\widetilde{\mathfrak{g}}_C)$, and use this to prove (4) *in $U(\widetilde{\mathfrak{g}}_C)$.*

Solution. We follow the suggested procedure and prove equation (5) by induction on $1 - c_{ij} \geq 1$. The base case $c_{ij} = 0$ is obvious. For the inductive step, we observe that

$$[E_i, E_i^t E_j E_i^{1-c_{ij}-t}] = E_i^t \overbrace{[E_i, E_j]}^{E_iE_j - E_jE_i} E_i^{1-c_{ij}-t} = E_i^{t+1} E_j E_i^{1-c_{ij}-t} - E_i^t E_j E_i^{1-c_{ij}+1-t}.$$

Thus, when expanding $[E_i, \mathrm{ad}_{E_i}^{1-c_{ij}}(E_j)]$ via (5), we get a sum where the coefficient of each term $E_i^{t+1} E_j E_i^{1-c_{ij}-t}$ is

$$(-1)^t \binom{1-c_{ij}}{t} - (-1)^{t+1} \binom{1-c_{ij}}{t+1} = (-1)^t \binom{1-c_{ij}+1}{t}.$$

This proves equation (5). In order to prove equation (4) we observe that the only nontrivial cases are $k = i, j$. If $k = j$, equation(5) and Liebnitz formula give

$$\left[F_j, \mathrm{ad}_{E_i}^{1-c_{ij}}(E_j)\right] = \sum_{t=0}^{1-c_{ij}} (-1)^t \binom{1-c_{ij}}{t} E_i^t H_j E_i^{1-c_{ij}-t} = \mathrm{ad}_{E_i}^{1-c_{ij}}(H_j),$$

and the relation of $\widetilde{\mathfrak{g}}_C$ tell us that the right hand side vanishes. Finally, let $k = i$ in (4). We work out the case $c_{i,j} = -1$ explicitely. Using (5) we have

$$\left[F_i, \mathrm{ad}_{E_i}^2(E_j)\right] = [F_i, E_i^2 E_j - 2E_iE_jE_i + E_jE_i^2]. \tag{6}$$

We expand the first summand by using relations (154)-(157) and the defining relations of the universal enveloping algebra. This gives

$$\begin{aligned}[F_i, E_i^2 E_j] = H_i E_i E_j + E_i H_i E_j = \\ (E_i H_i E_j - c_{i,i} E_i E_j) + E_i H_i E_j = \\ 2E_i E j H_i - c_{i,i} E_i E_j - 2c_{i,j} E_i E_j = 2E_i E j H_i.\end{aligned}$$

Performing analogous computations for each term of (6) gives $\left[F_i, \mathrm{ad}^2_{E_i}(E_j)\right] = 0$.

4. In the lecture, we showed how to associate to any Dynkin diagram X a complex semisimple Lie algebra $\mathfrak{g}_X$. Show that if a Dynkin diagram Y contains another Dynkin diagram X inside it (by which we mean that X contains as many edges between any two of its vertices as Y did), then there is an injective homomorphism $\mathfrak{g}_X \hookrightarrow \mathfrak{g}_Y$ between the corresponding Lie algebras.

Solution. Assume that X is irreducible and let C_X and C_Y be the Cartan matrices associated to X and Y respectively. The hypothesis that X is contained in Y is equivalent to saying that

$$C_Y = \begin{bmatrix} C_X & * \\ * & * \end{bmatrix}. \tag{7}$$

In turn, equation (7) tells us that the Lie algebra $\mathfrak{g}_Y$ has a subset of generators whose relations $(154) - (159)$ match those of the generators of $\mathfrak{g}_X$. Then we have a morphism $\mathfrak{g}_X \to \mathfrak{g}_Y$. The aforementioned morphism is necessarely injective since the Lie algebra $\mathfrak{g}_X$ is simple. If X is not irreducible, it is enough to run the above proof for all irreducible components of X.
As an example, observe that (3) can be extended to a homomorphism $\mathfrak{sl}_n \hookrightarrow \mathfrak{o}_{2n}$.

(*) With the notation from the lecture notes, prove that $\mathfrak{i}$ is contained in any ideal of $\widetilde{\mathfrak{g}}_C$ that has finite codimension (i.e. $\mathfrak{g}_C$ is the largest finite-dimensional quotient of $\widetilde{\mathfrak{g}}_C$).