Math 429 - Exercise Sheet 5 Solutions

1 Excercise 1

Express the adjoint representation of $\mathfrak{sl}_{2,\mathbb{C}}$ in terms of the irreducible representations $L(n)$ from the Lecture (try doing so as explicitly as possible).

Solution. The usual basis

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, F = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

is a basis of eigenvectors for the adjoint action of H on $\mathfrak{sl}_2$. In particular, we have

$$\mathrm{ad}_H(E) = 2E, \mathrm{ad}_H(H) = 0, \mathrm{ad}_H(F) = -2F.$$

The adjoint representation is irreducible and thus it is isomorphic to $L(2)$.

2 Excercise 2

Recall the isomorphism $\mathfrak{sl}_{2,\mathbb{C}} \cong \mathfrak{so}_{3,\mathbb{C}}$ from the previous exercise sheet. Express the standard 3-dimensional representation of $\mathfrak{sl}_{3,\mathbb{C}}$ in terms of the irreducible representations $L(n)$ from the Lecture (try doing so as explicitly as possible).

Solution. With the same notations as in the previous excercise sheet, we see that the isomorphisn $\mathfrak{sl}_{2,\mathbb{C}} \cong \mathfrak{so}_{3,\mathbb{C}}$ sends H to

$$-i2A = 2 \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix admits a basis of eigenvectors over the complex numbers,

$$\mathbb{C}^3 = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \mathbb{C} \oplus \begin{bmatrix} -1 \\ i \\ 0 \end{bmatrix} \mathbb{C} \oplus \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathbb{C}.$$

The first eigenvector has eigenvalue 2, the second one has eigenvalue -2, and the third vector is in the kernel of H. Furthermore, the isomorphism $\mathfrak{sl}_{2,\mathbb{C}} \cong \mathfrak{so}_{3,\mathbb{C}}$ sends F to

$$B + iC = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & i \\ 1 & -i & 0 \end{bmatrix}.$$

Then we can verify that

$$F \cdot \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, F \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \\ 0 \end{bmatrix}, F \cdot \begin{bmatrix} -1 \\ i \\ 0 \end{bmatrix} = 0.$$

Thus this representation is isomorphic to $L(2)$.

Note, however, that the isomorphism $\mathfrak{sl}_{2,\mathbb{C}} \cong \mathfrak{so}_{1,3}$ of real Lie algebras gives rise to a 4-dimensional real representation of $\mathfrak{sl}_{2,\mathbb{C}}$ which does not fit into the framework from Lecture, since the latter only applies to complex representations.

3 Complete reducibility for $\mathfrak{sl}_{2,\mathbb{C}}$

In what follows, we will show that any finite-dimensional complex representation $\mathfrak{sl}_{2,\mathbb{C}} \curvearrowright V$ is completely reducible, as long as H acts on V by a diagonalizable matrix (the latter condition holds due to the Jordan decomposition in semisimple Lie algebras, which we will study in Lecture 8).

Before going through the excercise, we state some concequences of the above assumptions.

Lemma 1. *Under our assumptions, E and F act as nilpotent endomorphisms. Moreover, all the eigenvalues of H are integers.*

Proof. Since H is diagonalizable, we have a decomposition of V (as a vector space)

$$V = \bigoplus_{l \in \mathbb{C}} V_l,$$

where $V_l = \{v \in V | Hv = lv\}$. The relations of $\mathfrak{sl}_2$ imply that $E(V_l) \subset V_{l+2}$ and $F(V_l) \subset V_{l-2}$. Since V is finite dimensional, we conclude that E and F are nilpotent.
Let v be an eienvector for H and let $l \in \mathbb{C}$ be the relative eigenvalue. Let $k+1$ be the smallest integer such that $F^{k+1}v = 0$, and consider the nonzero vectors

$$v_0 = v,\ v_1 = Fv, \dots, v_k = F^k v.$$

Then the relations of $\mathfrak{sl}_2$ yield

$$Hv_i = (l-2i)v_i \quad \text{and} \quad Ev_i = (l-i+1)iv_i.$$

Thus

$$0 = Ev_{k+1} = (l-(k+1)+1)(k+1)v_k$$

which implies $l = k \in \mathbb{Z}$. □

3.1 Excercise 3

Show that V is isomorphic (as a representation of $\mathfrak{sl}_{2,\mathbb{C}}$) to the direct sum of the generalized eigenspaces of the Casimir operator C

$$V = \bigoplus_{n \geq 0} \left\{ v \in V \middle| \left(C - \frac{n(n+2)}{2} \cdot I \right)^N (v) = 0 \text{ for some } N \gg 0 \right\}$$

Solution. We know from linear algebra that V has a decomposition into generalized eigenspaces for C

$$V = \bigoplus_{\lambda \in \mathbb{C}} \left\{ v \in V \middle| (C - \lambda \cdot I)^N (v) = 0 \text{ for some } N \gg 0 \right\}.$$

Since C commutes with $\mathfrak{sl}_2$ in $\mathrm{End}(V)$, it is easy to see that these generalized eigenspaces are in fact subrepresentations of V.
Thus it remains to show that the generalized eigenvalues of C are those prescribed by the statement. To this regard, suppose that $Cv = \lambda v$. Since E is nilpotent by Lemma 1, we can take $k+1$ to be the smallest integer such that $E^{k+1}v = 0$. Since E commutes with C, the vector $v' = E^k v$ satisfies again the eigenvector equation

$$Cv' = \lambda v'. \tag{1}$$

On the other side

$$Cv' = \left(2FE + H + \frac{H^2}{2}\right) v' = \left(H + \frac{H^2}{2}\right) v'. \tag{2}$$

Combining (1) and (2) we conclude that λ is an eigenvalue for $H + \frac{H^2}{2}$. Lemma 1 then implies that

$$\lambda = \frac{n(n+2)}{2},$$

where $n \in \mathbb{Z}$ is the H-eigenvalue associated to v'.

As a consequence, we henceforth restrict attention to proving the complete reducibility of a representation $\mathfrak{sl}_{2,\mathbb{C}} \curvearrowright V$ on which C has a single generalized eigenvalue, say $\frac{n(n+2)}{2}$.

Again, this assumption has an immediate consequence.

Lemma 2. *Under the above assumption, let $0 \neq v \in V$ be a vector such that $Ev = 0$. Then $Hv = nv$, and the subrepresentation of V generated by v is isomorphic to $L(n)$.*

Proof. Reasoning as in the proof of Lemma 1, we see that

$$\mathrm{Ker}(E) = \bigoplus_{l' \in \mathbb{Z}} V_l = V_n, \tag{3}$$

where the direct sum is performed over the eigenvalues l' of H such that $V_{l'} \neq 0$ and $V_{l'+2} = 0$. We see from the proof of Excercise 3 that for every such integer, $\frac{l'(l'+2)}{2}$ is an eigenvalue for C, and the second equality in (3) follows.
Let $\langle v \rangle$ be the subrepresentation of V generated by v. Elements in $\langle v \rangle$ are of the form

$$x \cdot v$$

where x is a polyniomial in E, F, H with complex coefficients. Since $Ev = 0$ and $Hv = nv$, such expression can be simplified to a complex linear combination of $v, Fv, \ldots, F^n v$, where n is the least integer such that $F^{n+1}v = 0$ because $v \in V_n$. Then $\langle v \rangle = \mathrm{Span}(v, Fv, \ldots, F^n v)$ is naturally isomorphic (as a representation) to $L(n)$. □

3.2 Excercise 4

Show that any irreducible sub or quotient representation of V as above is isomorphic to $L(n)$, hence the eigenvalues of H (which we assume to be diagonalizable) are $n, n-2, \ldots, 2-n, -n$.

Solution. An irreducible subrepresentation W od V is isomorphic to $L(m)$ where $m+1 = \dim W$. Let $0 \neq w \in W$ be such that $Ev = 0$. Then $Hw = mw$ by the definition of $L(m)$, and $Ew = nw$

by Lemma 2, thus $m = n$.
Similarly, let $V/W \cong L(m)$ be an irreducible quotient representation of V. The action of C on this quotient has the same generalized eigenvalue $\frac{n(n+2)}{2}$ as the action on V. Reasoning as above, this gives $m = n$.

3.3 Excercise 5

Show that V is completely reducible by induction on $\dim V$ and the following statement: any surjective $\mathfrak{sl}_{2,\mathbb{C}}$ intertwiner $g : V \twoheadrightarrow L(n)$ splits, i.e. $\exists \psi : L(n) \to V$ such that $g \circ \psi = \mathrm{Id}$.

Hint: note that $\mathrm{Ker}\ g \cong L(n)^{\oplus k}$ *for some* k*. It suffices to pick some eigenvector of* H *in* $V \backslash \mathrm{Ker}\ g$ *with eigenvalue* n*, and to show that it generates a subrepresentation of* V *isomorphic to* $L(n)$*.*

Solution. Observe that a surjective morphism $g : V \twoheadrightarrow L(n)$ exists. Indeed, if V is not irreducible, we can take a maximal proper subrepresentation $\{0\} \neq W \subset V$. Then the quotient V/W is irreducible and thus it is isomorphic to $L(n)$ by Excercise 4. By the inductive hypothesis $W = \mathrm{Ker}(g)$ is completely reducible, and Excercise 4 tells us that $W \cong L(n)^{\oplus k}$ for some integer k. Then we get an exact sequence of representations

$$0 \to L(n)^{\oplus k} \to V \to L(n) \to 0. \quad (4)$$

Finally, take a nonzero vector $v \in V - W$ such that $Ev = 0$, which exists since E is nilpotent. Lemma 2 implies that $\langle v \rangle \cong L(n)$. Moreover, $\langle v \rangle \cap W = \{0\}$ since the intersection of subrepresentations is a subrepresentation, and $\langle v \rangle$ is irreducible. This proves that the exact sequence (4) splits, and the splitting lemma concludes.