Math 429 - Exercise Sheet 13

1. Prove Proposition 32.

Solution. We know that the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ act in a semisimple way, so we have a decomposition $V = \bigoplus_\lambda V_\lambda$ as in equation (172) of the Lecture notes. We have to prove that

$$\frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z} \tag{1}$$

for every weight λ and for every simple root α_i. Let $\mathfrak{sl}_2 \cong \langle F_{\alpha_i}, H_{\alpha_i}, E_{\alpha_i} \rangle$ be the triple associated to α_i. If $v \in V_\lambda$, then

$$H_{\alpha_i} v = \lambda(H_{\alpha_i}) v = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} v.$$

The theory on representations of $\mathfrak{sl}_2$ implies (1).

2. Prove Proposition 34.

Solution. Both sides of the homomorphism

$$\pi \colon M(\lambda) \twoheadrightarrow L(\lambda)$$

from section 13.4 in the Notes are cyclic modules. It is then clear that this map is surjective. As mentioned in the Lecture notes, $M(\lambda)$ has a unique maximal graded subrepresentation, and the claim follows.

3. Prove that for any $\lambda, \mu \in \mathfrak{h}^*$, there exists an injective $\mathfrak{g}$-intertwiner

$$L(\lambda + \mu) \hookrightarrow L(\lambda) \otimes L(\mu)$$

(where $L(\lambda)$ denotes the irreducible representation of $\mathfrak{g}$ with highest weight λ, as in the notes).

Solution. We know that the $\mathfrak{g}$ module $L(\lambda) \otimes L(\mu)$ is completely reducible. Write

$$L(\lambda) \otimes L(\mu) = L(\lambda + \mu)^{\oplus N} \bigoplus_{\nu < \lambda + \mu} L(\nu), \tag{2}$$

where $\lambda + \mu$ is the highest weight and the sum is on all ν in the weight lattice such that $(\lambda + \mu) - \nu$ is a sum of positive roots with non negative coefficients. The highest weight vector $(1 \otimes v_\lambda) \otimes (1 \otimes v_\mu)$ has to sit in a summand of (2), thus $N \geq 1$.

4. Work out the weight decomposition of the symmetric power representation $V = S^k \mathbb{C}^n$ of $\mathfrak{sl}_n$, i.e. write down those $\lambda \in P$ for which $V_\lambda \neq 0$.

Solution. Consider the usual base

$$S^k \mathbb{C}^n = \bigoplus_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \mathbb{C}(v_{i_1} \otimes \cdots \otimes v_{i_k}). \tag{3}$$

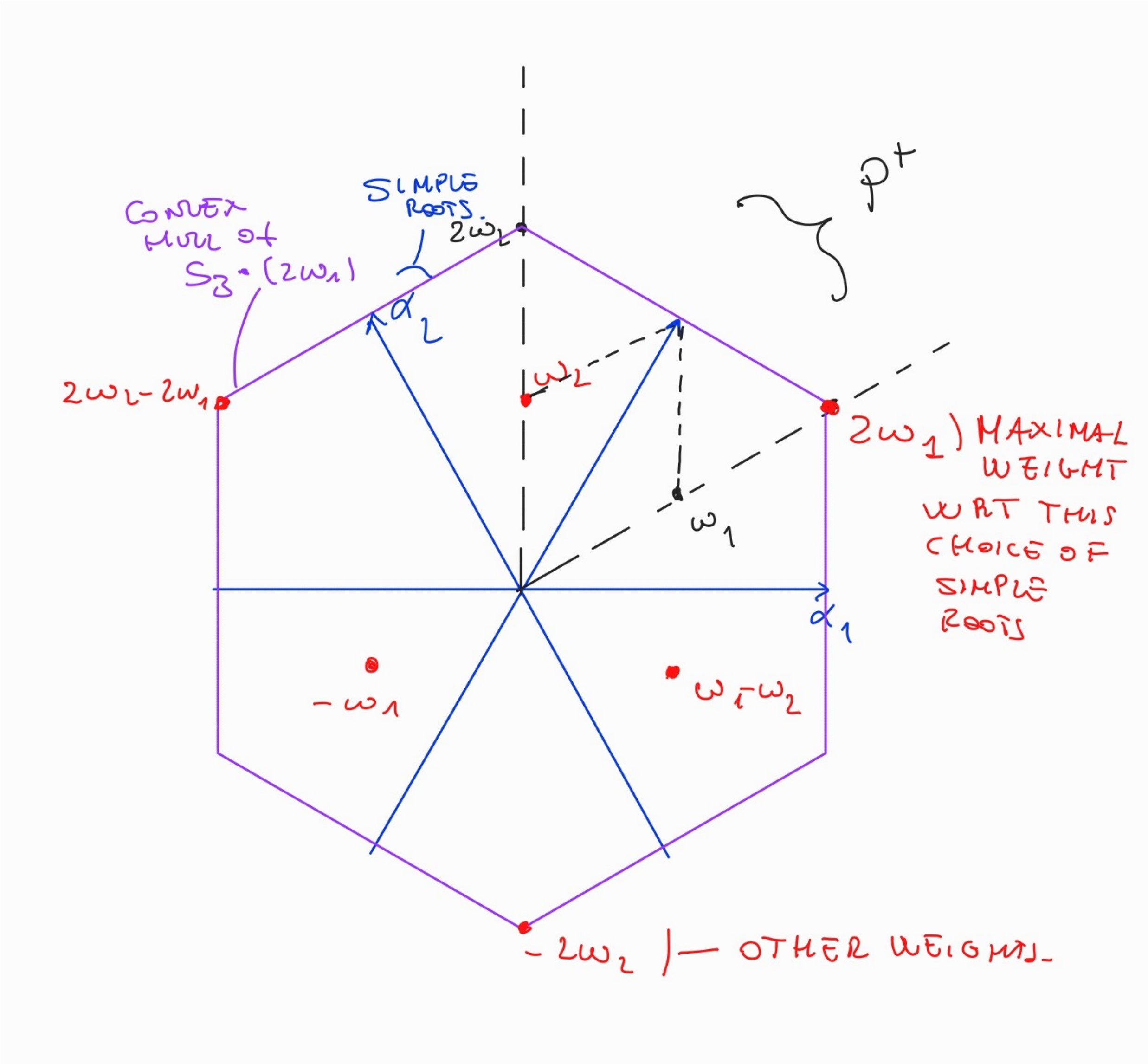
Every trace - zero diagonal matrix $h = (x_1, \ldots, x_n)$ in the Cartan subalgebra of $\mathfrak{sl}_n$ acts on each of these basis element as

$$(x_1, \ldots, x_n) \cdot (v_{i_1} \otimes \cdots \otimes v_{i_k}) = (x_{i_1} + \cdots + x_{i_k})(v_{i_1} \otimes \cdots \otimes v_{i_k}).$$

Then we see that (3) is already the weight decomposition and all weight spaces are one dimensional. In terms of the abstract root system associated to $\mathfrak{sl}_n$, the weights are

$$e_{i_1} + \cdots + e_{i_k}. \tag{4}$$

We display the theory by describing explicitely the weight decomposition of the action of $\mathfrak{sl}_3$ on $S^2\mathbb{C}^3$. You can check that in this case the weights (4) are those in the following picture.



Here, $\alpha_i = e_i - e_{i+i}$ denote the simple roots, and ω_i stands for the fundamental weights in Example 10 of the Lecture Notes. Observe that the set of red weights in the picture is invariant with respect to the Weyl group action.

In fact, we have a criterion for an integral weight μ to occur in the root decomposition of the module $L(\lambda)$.

Theorem 1. *Let $L(\lambda)$ be a finite-dimensional, irreducible representation for the complex semisimple Lie algebra $\mathfrak{g}$. An element μ of the integral weight lattice is a weight for $L(\lambda)$ if and only if*

1. *μ belongs to the convex hull of the Weyl group orbit of λ,*
2. *$\lambda - \mu$ is a sum of positive roots with non-negative integer coefficients.*

In the case at hand, it is easy to see that the highest weight vector is $v_1 \otimes v_1$, and the highest weight is $2\omega_1$. As an example, let us verify that the weight ω_2 satisfies condition 2 in the above theorem. Using the definitions of the fundamental weights, we have

$$2\omega_1 - \omega_2 = 2\left(\frac{1}{3}\alpha_1 + \frac{1}{3}(\alpha_1 + \alpha_2)\right) - \left(\frac{1}{3}(\alpha_1 + \alpha_2) + \frac{1}{3}\alpha_2\right) = \alpha_1.$$

5. Consider the tautological representation of $\mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}$ respectively (namely $\mathbb{C}^{2n+1}, \mathbb{C}^{2n}, \mathbb{C}^{2n}$ respectively). Determine the highest weight of these three representations; recall our conventions on the root systems of types B, C, D and their simple roots in Ex. 1, Sheet 10 and Ex. 1, Sheet 11.

Solution. Let us consider $\mathfrak{o}_{2n}$. Recall the choices of Cartan subalgebra from Sheet 9, and positive and simple roots from Sheet 10. The vector

$$(1\ i\ 0\ \dots\ 0)^T \in \mathbb{C}^{2n}$$

is annihilated by every upper triangular matrix and it is a eigenvalue for every matrix

$$\begin{bmatrix} 0 & a_1 & & & \\ -a_1 & 0 & & & \\ & & 0 & a_2 & \\ & & -a_2 & 0 & \\ & & & & \ddots \end{bmatrix}$$

with eigenvalue ia_1. In the previous notations, this eigenvalue corresponds to the linear map $i\alpha_1$ in the dual of the Cartan subalgebra. In terms of the associated abstract root system of Sheet 10, this corresponds to the element $e_1 \in \mathbb{R}^n$. It is easy to check that e_1 is a dominant integral weight. Indeed, for every $1 \leq k \leq n-1$

$$(e_1, e_k - e_{k+1}) = \delta_{1,k},$$

and $(e_1, e_{n-1} + e_n) = 0$.

(*) Let V be a representation of a finite-dimensional <u>abelian</u> Lie algebra $\mathfrak{h}$, which admits a weight decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \qquad \text{where } V_\lambda = \{v \in V | x \cdot v = \lambda(x)v, \forall x \in \mathfrak{h}\}$$

If the weight spaces V_λ are all finite-dimensional, then show that any subrepresentation and quotient representation of V also admits a weight decomposition.