Math 429 - Exercise Sheet 10

1. In the previous exercise sheet, you worked out the root decompositions of the complex semisimple Lie algebras of types A, B, C, D. In each of those cases, write out all the roots as linear combinations of a fixed choice of simple roots.

Solution. Consider the root system A_{n+1}, with the decomposition from Lecture 10

$$R^+ = \{e_i - e_j \,\Big|\, 1 \leq i < j \leq n\} \quad R^- = \{e_i - e_j \,\Big|\, 1 \leq j < i \leq n\}.$$

The simple roots are $\alpha_i = e_i - e_{i+1}$ for $i = 1, \dots, n$. Then we can write positive roots as

$$e_i - e_j = \alpha_i + \cdots + \alpha_{j-1}$$

for every $1 \leq i < j \leq n$. The decomposition of negative roots follows.

Recall the root system D_n associated to the Lie algebra $\mathfrak{o}_{2n}$ from excercise sheet 9. The notation $A = (a_1 \dots, a_n)$ stands for the 2×2 block-diagonal matrices whose kth block is

$$\begin{bmatrix} 0 & a_k \\ -a_k & 0 \end{bmatrix}, \tag{1}$$

for some $a_k \in \mathbb{C}$. Let $\mathfrak{h}$ be the toral subalgebra which consists of all matrices $A = (a_1, \dots, a_n)$ as above. A basis for the dual $\mathfrak{h}^*$ is given by the linear maps $\alpha_k \colon A = (a_1, \dots, a_n) \mapsto a_k$, and we found the $2n(n-1)$ roots

$$\{i(\pm\alpha_k \pm' \alpha_l)\} \subset \mathfrak{h}^*. \tag{2}$$

We can see the subset (2) as a root system by means of the non-degenerate product $(A, B) \mapsto \mathrm{tr}(AB)$ on $\mathfrak{h}$. More precisely, the $\mathbb{R}$-vector space generated by the roots (2) is isometric to the standard euclidean space $\mathbb{R}^n$ via the identification

$$i\alpha_k \mapsto \frac{1}{2}e_k.$$

Thus after rescaling by a factor of 2 we get the root system $\{\pm e_k \pm^{‘} e_l\} \subset \mathbb{R}^n$. We make the following choices of positive roots

$$R^+ = \{e_k - e_l, e_k + e_l \,\Big|\, 1 \leq k < l \leq n\} \tag{3}$$

and of simple roots

$$\beta_k = e_k - e_{k+1},\ k = 1, \dots, n-1, \quad \beta_n = e_{n-1} + e_n.$$

Then the positive roots admit the following decomposition

$$\begin{aligned} e_k - e_l &= \beta_k + \cdots + \beta_{l-1}, \\ e_k + e_n &= (e_k - e_{n-1}) + (e_{n-1} + e_n) = \beta_k + \cdots + \beta_{n-2} + \beta_n, \\ e_k + e_l &= (e_k + e_n) + (e_l - e_n) = \beta_k + \cdots + \beta_{n-2} + \beta_n + \beta_l + \cdots + \beta_{n-1}, \end{aligned}$$

and the decomposition of negative roots follows.

We proceed as above. The roots of the Lie algebra $\mathfrak{o}_{2n+1}$ are identified with the root system

$$\{\pm e_k \pm' e_l, \pm e_k \,\Big|\, 1 \le k < l \le n\} \subset \mathbb{R}^n. \tag{4}$$

The positive roots are

$$R^+ = \{e_k - e_l, e_k + e_l, e_k \,\Big|\, 1 \le k < l \le n\},$$

and the simple roots are

$$\beta_k = e_k - e_{k-1},\; k = 1, \dots, n-1, \quad \beta_n = e_n.$$

Then the positive roots admit the following decomposition

$$\begin{aligned} e_k - e_l &= \beta_k + \cdots + \beta_{l-1}, \\ e_k = (e_k - e_n) + e_n &= \beta_k + \cdots + \beta_{n-1} + \beta_n, \\ e_k + e_l &= \beta_k + \cdots + \beta_n + \beta_l + \cdots + \beta_n, \end{aligned}$$

and the decomposition of negative roots follows.

Finally we carry out the same computation for the root system C_n associated to the Lie algebra $\mathfrak{sp}_{2n}$. We have the root system

$$\{\pm e_k \pm' e_l, \pm 2e_k \,\Big|\, 1 \le k < l \le n\} \subset \mathbb{R}^n. \tag{5}$$

The positive roots are

$$R^+ = \{e_k - e_l, e_k + e_l, 2e_k \,\Big|\, 1 \le k < l \le n\},$$

and the simple roots are

$$\beta_k = e_k - e_{k-1},\; k = 1, \dots, n-1, \quad \beta_n = 2e_n.$$

Then the positive roots admit the following decomposition

$$\begin{aligned} e_k - e_l &= \beta_k + \cdots + \beta_{l-1}, \\ 2e_k = (2e_k - 2e_n) + 2e_n &= 2\beta_k + \cdots + 2\beta_{n-1} + \beta_n, \\ e_k + e_l &= 2\beta_k + \cdots + 2\beta_{n-1} + \beta_n + 2\beta_l + \cdots + 2\beta_{n-1} + \beta_n, \end{aligned}$$

and the decomposition of negative roots follows.

2. Show that the following subset of $\mathbb{R}^n$

$$R = \Big\{ \pm e_i, \pm 2e_i \Big\}_{1 \le i \le n} \sqcup \Big\{ \pm e_i \pm' e_j \Big\}_{1 \le i < j \le n}$$

determines a **non-reduced** root system, i.e. satisfies all axioms in Definition 19, except for (136) (specifically, if α is a root, then we do allow 2α or $\frac{\alpha}{2}$ to be a root). It is called "type BC_n".

Solution. The set R clearly generates $\mathbb{R}^n$. Moreover most of axioms (137) and (138) in Definition 19 follow from the same statements for the root systems B_n and C_n. We only have to check that

$$c_{e_i, 2e_j} = \frac{2(e_i, 2e_j)}{(e_i, e_i)} = 4\delta_{i,j}, \text{ and } c_{2e_j, e_i} = \frac{2(e_i, 2e_j)}{(2e_j, 2e_j)} = \delta_{i,j},$$

are integers, and that

$$2e_j - c_{e_i,2e_j}e_i = 2e_j - 4\delta_{i,j}e_i, \text{ and } e_i - c_{2e_j,e_i}2e_j = e_i - \delta_{i,j}2e_j$$

belong to R. Both claims are obvious.

3. Consider the root system of type D_6 with simple roots $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3 - e_4$, $\alpha_4 = e_4 - e_5$, $\alpha_5 = e_5 - e_6$, $\alpha_6 = e_5 + e_6$, and consider the following decomposition

$$\mathbb{R}^6 = \text{span}\Big(\alpha_4 - \varphi\alpha_2, \alpha_6 - \varphi\alpha_1, \alpha_3 - \varphi\alpha_5.\Big) \oplus \text{span}\Big(\varphi\alpha_4 + \alpha_2, \varphi\alpha_6 + \alpha_1, \varphi\alpha_3 + \alpha_5\Big)$$

where $\varphi = \frac{1+\sqrt{5}}{2}$. Show that the three-dimensional subspaces in the right-hand side are orthogonal. If we let $\pi : \mathbb{R}^6 \to \mathbb{R}^3$ denote the orthogonal projection onto the second subspace, show that

$$\pi(60 \text{ roots of } D_6) = R \sqcup \varphi R \tag{6}$$

for some $R \subset \mathbb{R}^3$ of cardinality 30. This set R determines a **non-crystallographic** root system, i.e. satisfies all axioms in Definition 19 except for (137). It is called "type H_3" and it is related to symmetries of a regular icosahedron.

Solution. In order to prove that the two given subspaces are orthogonal it is enough to check the scalar products of the generators. For instance

$$\langle \alpha_4 - \varphi\alpha_2, \varphi\alpha_6 + \alpha_1 \rangle = \varphi - \varphi = 0,$$

and the other relations are analogous. We now turn to the second claim. An easy computation shows that

$$\pi(\alpha_4) = \varphi\left(\frac{1}{\varphi^2+1}(\varphi\alpha_4 + \alpha_2)\right) = \varphi\pi(\alpha_2).$$

Similarly $\pi(\alpha_6) = \varphi\pi(\alpha_1)$, and $\pi(\alpha_3) = \varphi\pi(\alpha_5)$. One can list all the 30 positive roots (3) from Exercise 1 and verify that they split into two disjoint subsets, according to (6). Finally, R inherits the properties of a non-crystallographic root system from D_6.

4. The length of an element w in the Weyl group W (of a root system R with a given set of simple roots I) is defined as

$$\ell(w) = \min\Big\{k \geq 0 \Big| \exists i_1, \ldots, i_k \in I \text{ s.t. } w = s_{i_1} \ldots s_{i_k}\Big\}$$

Show that $\ell : W \to \mathbb{Z}_{\geq 0}$ is a length function, i.e.

$$\ell(e) = 0$$

$$\ell(w^{-1}) = \ell(w)$$

$$\ell(w_1w_2) \leq \ell(w_1) + \ell(w_2)$$

Show that there exists a unique element of W of maximal length (how does it act on roots and on Weyl chambers?) What is the length function for the type A_{n-1} root system, for which $W = S_n$?

Solution. Showing that $\ell : W \to \mathbb{Z}_{\geq 0}$ is a length function is immediate. Indeed observing $w = s_{i_1} \ldots s_{i_k}$ if and only if $w^{-1} = s_{i_k} \ldots s_{i_1}$ shows that $\ell(w^{-1}) = \ell(w)$. Similarly, if $w_1 = s_{i_1} \ldots s_{i_k}$ and $w_2 = s_{i'_1} \ldots s_{i'_k}$, then $w_1w_2 = s_{i_1} \ldots s_{i_k}s_{i'_1} \ldots s_{i'_k}$, so that $\ell(w_1w_2) \leq \ell(w_1) + \ell(w_2)$.
Since the Weyl group is finite, we only have to prove that the element of maximal length is unique. The following is proven in Humphreys'book.

Theorem 1. *The length of an element $w \in W$ can be expressed as*

$$\ell(w) = \textit{number of positive roots } \alpha \textit{ such that } w(\alpha) \textit{ is negative.}$$

As a corollary, we find that the length function attains its maximum only in the element $w_0 \in W$ which changes sign to every root

$$w_0 = \prod_{\alpha \text{ simple root}} s_\alpha.$$

As a remark we observe that Theorem 1 can be rephrased as follows. According to the proof of Proposition 25 in the Lecture Notes, a decomposition of an element $w \in W$ into simple reflections

$$w = s_{i_1} \dots s_{i_k}$$

is equivalent to a sequence of Weyl chambers $\mathcal{C}_0 = \mathcal{C}^+, \mathcal{C}_1, \dots, \mathcal{C}_k$ such that

$$\mathcal{C}_l = s_{i_1} \dots s_{i_l}(\mathcal{C}^+).$$

Then, the longest element in the Weyl group has to be the one which sends the positive chamber $\mathcal{C}^+$ to the negative chamber.
For the root system A_{n-1}, the isomorphism $W = S_n$ depends on the choice of an ordering on the set of simple roots I, and sends the ith simple root to the transposition $(i, i+1)$. Hence, this isomorphism identifies the length function in W with the function on S_n which counts the minimum number of transpositions.

(*) If $\mathfrak{g}$ is a (not necessarily semisimple) Lie algebra over a field of characteristic 0, a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a Cartan subalgebra if it is nilpotent and self-normalizing:

$$\left\{ x \in \mathfrak{g} \middle| [x, \mathfrak{h}] \subseteq \mathfrak{h} \right\} = \mathfrak{h}$$

Prove that any Cartan subalgebra is a maximal nilpotent subalgebra (the converse fails in general, though, think about the subalgebra $\mathfrak{n} \oplus \mathbb{C}I_n$ of $\mathfrak{gl}_n$).