Math 429 - Exercise Sheet 8

1. Calculate the Killing form of $\mathfrak{sl}_n$.

Solution. Recall from Excercises 4 and 5 of Sheet 7 that the Killing form of $\mathfrak{sl}_n$ is $2n$ times the trace form. Then it is enough to consider the basis

$$(E_{i,j})_{i\neq j, 1\leq i,j\leq n} \cup (H_i = E_{i,i} - E_{nn})_{1\leq i\leq n-1} \tag{1}$$

and compute the trace form. We have

$$\operatorname{tr}(E_{i,j}E_{k,l}) = \operatorname{tr}(\delta_{j,k}E_{i,l}) = \delta_{j,k}\delta_{i,l} = \begin{cases} 1 & (k,l) = (j,i) \\ 0 & \text{otherwise.} \end{cases}$$

Similar computations yield $\operatorname{tr}(E_{i,j}H_k) = 0$ for all $i \neq j$, k, and

$$\operatorname{tr}(H_iH_j) = \begin{cases} 2 & i = j \\ 1 & i \neq j. \end{cases}$$

In particular, we observe that the Killing form of $\mathfrak{sl}_n$ is nondegenerate.

2. Let $\mathfrak{p} \subset \mathfrak{sl}_{m+n}$ be the **parabolic** subalgebra consisting of matrices of the form

$$\begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

where A and B are traceless $m \times m$ and $n \times n$, respectively. Calculate $\operatorname{rad}(\mathfrak{p})$ and $\mathfrak{p}_{ss} = \mathfrak{p}/\operatorname{rad}(\mathfrak{p})$.

Solution. Consider the ideal $\mathfrak{i}$ consisting of matrices of the form

$$\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}.$$

Lemma 3 in the lecture notes implies that $\mathfrak{i}$ is solvable. Moreover, the quotient $\mathfrak{p}/\mathfrak{i}$ is isomorphic to the semisimple Lie algebra $\mathfrak{sl}_n \oplus \mathfrak{sl}_m$. This implies that $\mathfrak{i}$ is not properly contained in any solvable ideal of $\mathfrak{p}$.

3. Calculate the Casimir element of $\mathfrak{o}_3$ and its action on the tautological 3-dimensional representation of $\mathfrak{o}_3$.

Solution. The matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

form a basis of $\mathfrak{o}_3$, with relations

$$[A,B] = C, \ [B,C] = A, \ [C,A] = B. \tag{2}$$

In order to find the Casimir element with respect to the basis (A, B, C), we need to compute a dual basis[1] with respect to the Killing form. Relations (2) yield

$$\mathrm{ad}_A = -C, \ \mathrm{ad}_B = -B, \ \mathrm{ad}_C = -A,$$

so one can verify that $(-\frac{1}{2}A, -\frac{1}{2}B, -\frac{1}{2}C)$ is the dual basis we are looking for. We conclude that the Casimir alement is given by

$$c = -\frac{1}{2}(A^2 + B^2 + C^2).$$

Finally, this Casimir element acts as a scalar matrix in any irreducible representation. Then, the equality $cA = A$ implies that c acts as the identity in the standard representation of $\mathfrak{o}_3$.

4. If X is a diagonalizable $n \times n$ matrix, prove that

$$\mathrm{ad}_X : \mathfrak{gl}_n \to \mathfrak{gl}_n, \qquad \mathrm{ad}_X(Y) = [X, Y]$$

is also diagonalizable.

Solution. After a change of basis, we can assume that X is diagonal. Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{K}^n$, so that $Xe_i = \lambda_i e_i$ for all i. We prove that $(E_{i,j})_{1 \le i,j \le n}$ is a basis if eigenvector for the endomorphism ad_X. Indeed the computation

$$\begin{aligned}\mathrm{ad}_X(E_{i,j})e_k = XE_{i,j}e_k - E_{i,j}Xe_k = X\delta_{j,k}e_i - E_{i,j}\lambda_k e_k = \\ (\lambda_j - \lambda_k)\delta_{j,k}e_i = (\lambda_j - \lambda_k)(E_{i,j}e_k)\end{aligned}$$

shows that $\mathrm{ad}_X(E_{i,j}) = (\lambda_j - \lambda_k)E_{i,j}$.

5. If we assume that a $n \times n$ complex matrix X is conjugate to a direct sum of Jordan blocks, then

- explicitly construct a diagonalizable matrix X_{ss} and a nilpotent matrix X_n such that

$$X = X_{ss} + X_n \tag{3}$$

- show that $X_{ss}X_n = X_nX_{ss}$
- show X_{ss} and X_n are complex polynomials in X with zero constant term
- show that the decomposition (3) is unique with respect to the properties above.

Solution. Suppose that the characteristic polynomial of X is $(t - \lambda_1)^{n_1}(t - \lambda_2)^{n_2} \cdots (t - \lambda_k)^{n_k}$, where $\lambda_1, \dots, \lambda_k$ are different complex numbers. Then, $\mathbb{C}^n$ splits as a sum of generalized eigenspaces

$$V_i = \{v \in \mathbb{C}^n s.t. (X - \lambda_i I_n)^N v = 0, N >> 0\}.$$

After a change of basis, we can assume that

$$X = \begin{bmatrix} J_{n_1}(\lambda_1) & & & \\ & J_{n_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{n_k}(\lambda_k) \end{bmatrix},$$

[1] Recall that (x_i) and (x^i) are dual basis for the nondegenerate bilinear form ϕ if $\phi(x_i, x^j) = \delta_{i,j}$.

where $J_{n_i}(\lambda_i)$ is the $n_i \times n_i$ Jordan block

$$\begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix}$$

associated to the generalized eigenspace V_i. Then it is clear that the matrices

$$X_{ss} = \begin{bmatrix} \lambda_1 I_{n_1} & & & \\ & \lambda_2 I_{n_2} & & \\ & & \ddots & \\ & & & \lambda_k I_{n_k} \end{bmatrix} \text{ and } X_n = \begin{bmatrix} J_{n_1}(0) & & & \\ & J_{n_2}(0) & & \\ & & \ddots & \\ & & & J_{n_k}(0) \end{bmatrix}$$

satisfy $X = X_{ss} + X_n$ and $X_{ss}X_n = X_n X_{ss}$. We now prove the third point. Since the numbers $\lambda_1, \dots, \lambda_k$ are different, we can apply the Chinese Reminder Theorem to find a polynomial $p \in \mathbb{C}[t]$ such that

$$\begin{cases} p(t) \equiv 0 \mod t \\ p(t) \equiv \lambda_i \bmod (t - \lambda_i)^{n_i} \quad \text{for all } i. \end{cases} \tag{4}$$

This conditions imply that $p(t)$ has zero constant term and that $p(X)\Big|_{V_i} = \lambda_i I_{n_i}$ for all i, hence $p(X) = X_{ss}$. Moreover, the polynomial $q(t) = t - p(t)$ has zero constant term and satisfies $q(X) = X_n$.

Finally we prove uniqueness. Suppose that we have another decomposition $X = S + N$ satisfying the first three points of the statement. In particular, the second point implies that every endomorphism X_{ss}, X_n, S, N commutes with eachother. In particular $X_{ss} - S$ is a sum of commuting semisimple operators, and hence it is a semisimple operator. Similarly $N - X_n$ is nilpotent. Thus, the equality $X_{ss} - S = N - X_n$ forces both sides to be 0.

(*) Prove the following analogue of the claim at the beginning of the proof of Theorem 17. For any Lie algebra $\mathfrak{g}$, consider its Lie algebra of derivations

$$\mathrm{Der}(\mathfrak{g}) \subseteq \mathrm{End}(\mathfrak{g})$$

as in Subsection 8.7. Show that the semisimple and nilpotent part of any $\zeta \in \mathrm{Der}(\mathfrak{g})$ (calculated as linear transformations of $\mathfrak{g}$) also lie in $\mathrm{Der}(\mathfrak{g})$.