Math 429 - Exercise Sheet 6

1. Show that a one-dimensional representation of any Lie algebra $\mathfrak{g}$ is the same as a covector of

$$\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$$

(i.e. a linear map from the above vector space to the ground field).

Solution. A one dimensional representation of $\mathfrak{g}$ is a Lie algebra homomorphism $\rho\colon \mathfrak{g} \to \mathbb{K}$. In particular, ρ is linear and $\rho([x,y]) = [\rho(x),\rho(y)] = 0$ for all $x, y \in \mathfrak{g}$. This means that ρ induces a linear map from $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ to the ground field $\mathbb{K}$. Viceversa, consider a covector $\psi\colon \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \to \mathbb{K}$. Denoting with $[x]$ the image of $x \in \mathfrak{g}$ in the quotient $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$, we can define a representation as above by setting $\rho(x) = \psi([x])$ for all $x \in \mathfrak{g}$.

2. Use the Poincaré-Birkhoff-Witt theorem to show that the universal enveloping algebra $U\mathfrak{g}$ of any Lie algebra $\mathfrak{g}$ (over any field) has no zero-divisors.

Solution. Let $y, z \in U\mathfrak{g}$ such that $yz = 0$. Denote as $\overline{y}$ and $\overline{z}$ the images of y and z in $\operatorname{gr} U\mathfrak{g}$, and write as $\overline{y} = (\overline{y})_0 + \cdots + (\overline{y})_l$ and $\overline{z} = (\overline{z})_0 + \cdots + (\overline{z})_k$ the associated decompositions with respect to the grading. The assumption $yz = 0$ in $U\mathfrak{g}$ implies that $\overline{yz} = 0$ in $\operatorname{gr} U\mathfrak{g}$, and in particular $0 = (\overline{yz})_{l+k} = (\overline{y})_l(\overline{z})_k$. However $(\overline{y})_l$ and $(\overline{z})_k$ are nonzero, which yields a contraddiction, since the algebra $\operatorname{gr} U\mathfrak{g}$ has no zero divisors by the PBW theorem.

We can solve the excercise without invoking the PBW theorem by writing down explicitely the above argument. Let $y, z \in U\mathfrak{g}$ such that $yz = 0$. With the same notation as in the lecture notes, suppose $z = z_1 \otimes \cdots \otimes z_k \in U_k\mathfrak{g}$ and $y = y_1 \otimes \cdots \otimes y_l \in U_l\mathfrak{g}$, for elements $z_i, y_i \in \mathfrak{g}$. One can expand the tensor products by writing the z_i's and y_i's with respect to a basis $(x_1, \ldots, x_n)$ of $\mathfrak{g}$. Then, we apply the procedure in the proof of the PBW theorem, and get the expansion of z and y in the PBW basis associated to $(x_1, \ldots, x_n)$. Explicitely,

$$\begin{aligned} y &= \sum_{i_1+\cdots+i_n=l} a_{i_1,\ldots,i_n} x_1^{i_1} \otimes \cdots \otimes x_n^{i_n} + [\text{terms of order } < l], \\ z &= \sum_{j_1+\cdots+j_n=k} b_{j_1,\ldots,j_n} x_1^{j_1} \otimes \cdots \otimes x_n^{j_n} + [\text{terms of order } < k], \end{aligned} \tag{1}$$

where $a_{i_1,\ldots,i_n}, b_{j_1,\ldots,j_n} \in \mathbb{K}$ and we may assume that not all of them are zero. When computing the product yz, we can perform the same procedure and get

$$yz = \sum_{\substack{i_1+\cdots+i_n=l,\\ j_1+\cdots+j_n=k}} a_{i_1,\ldots,i_n} b_{j_1,\ldots,j_n} x_1^{i_1+j_1} \otimes \cdots \otimes x_n^{i_n+j_n} + [\text{terms of order} < l+k]. \tag{2}$$

The assumption $yz = 0$ implies that the term of highest degree in (2) vanishes, which is a contraddiction. Observe that the top degree parts in the expansions (1) and (2) correspond to $(\overline{y})_l$, $(\overline{z})_k$, and $(\overline{yz})_{l+k}$respectively, and the procedure shows that the product in $\operatorname{gr} U\mathfrak{g}$ correspond to the product in $S\mathfrak{g}$.

3. Suppose we have a Lie algebra $\mathfrak{g}$ over a field of characteristic zero. While the assignment

$$S\mathfrak{g} \to U\mathfrak{g}, \qquad x_1 \ldots x_n \to \frac{1}{n!} \sum_{\sigma \in S(n)} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$$

is not an algebra homomorphism, show that it is a isomorphism of (infinite-dimensional) representations of $\mathfrak{g}$. *First you'll have to make $S\mathfrak{g}$ and $U\mathfrak{g}$ into representations of $\mathfrak{g}$: use the adjoint representation and equation (31) in the lecture notes.*

Solution. Let $x \in \mathfrak{g}$ and $y = y_1 \otimes \cdots \otimes y_l \in T\mathfrak{g}$. According to equation (31) in the lecture notes, the adjoint action induced by $\mathfrak{g}$ on $T\mathfrak{g}$ is

$$x \cdot y = [x, y_1] \otimes y_2 \cdots \otimes y_l + y_1 \otimes [x, y_2] \otimes \cdots \otimes y_l + \cdots + y_1 \otimes \cdots \otimes [x, y_l].$$

This action descends on both quotients $S\mathfrak{g}$ and $U\mathfrak{g}$. Let us fix a basis $(x_1, \ldots, x_n)$ of $\mathfrak{g}$, and in the following we will denote as

1. $[x_i, x_j] = \sum_k a_k^{i,j} x_k$ the sturcture constants,
2. $x_{1_1} \ldots x_{i_m}$ for $i_1 \leq \cdots \leq i_m$ the associated basis for $S\mathfrak{g}$,
3. $x_{i_1} \otimes \cdots \otimes x_{i_m}$ for $i_1 \leq \cdots \leq i_m$ the associated PBW basis of $U\mathfrak{g}$, and
4. $(x_{i_1} \cdots x_{i_m})^{\mathrm{sym}} = \frac{1}{m!} \sum_{\sigma \in S(m)} x_{\sigma(i_1)} \otimes \cdots \otimes x_{\sigma(i_m)}$ the map in the text.

Observe that the bijection between the basis 2. and 3. does not give an isomorphism of $\mathfrak{g}$ modules (for example, we may assume that $n = 3$ and verify that $x_2 \cdot (x_1 \otimes x_2) \neq x_2 \cdot (x_1 x_2)$, whereas

$$x_2 \cdot \frac{1}{2}(x_1 \otimes x_2 + x_2 \otimes x_1) = (x_2 \cdot x_1 x_2)^{\mathrm{sym}}).$$

The association $x_{i_1} \ldots x_{i_n} \mapsto (x_{i_1} \cdots x_{i_m})^{\mathrm{sym}}$ is a bijection between basis of two vector spaces and, so it extendes to a linear isomorphism. In order to prove that such linear isomorphism is an isomorphism of $\mathfrak{g}$ modules, we verify that

$$x_j \cdot (x_{i_1} \ldots x_{i_m})^{\mathrm{sym}} = (x_j \cdot x_{i_1} \ldots x_{i_m})^{\mathrm{sym}}. \tag{3}$$

The left hand side in (3) is

$$x_j \cdot \left(\frac{1}{m!} \sum_{\sigma \in S(m)} x_{\sigma(i_1)} \otimes \cdots \otimes x_{\sigma(i_m)} \right) = \tag{4}$$
$$\frac{1}{m!} \sum_{\sigma \in S(m)} [x_j, x_{\sigma(i_1)}] \otimes \cdots \otimes x_{\sigma(i_m)} + \cdots + x_{\sigma(i_1)} \otimes \cdots \otimes [x_j, x_{\sigma(i_m)}].$$

The right hand side in (3) is

$$([x_j, x_{i_1}] x_{i_2} \ldots x_{i_m})^{\mathrm{sym}} + \cdots + (x_{i_1} x_{i_2} \ldots [x_j, x_{i_m}])^{\mathrm{sym}}. \tag{5}$$

Let us expand the kth term in the above sum

$$(x_{i_1} \ldots [x_j, x_{i_k}] \ldots x_{i_m})^{\mathrm{sym}} = \left(x_{i_1} \ldots \left(\sum_{l=1}^{n} a_l^{j,i_k} x_l \right) \ldots x_{i_m} \right)^{\mathrm{sym}} =$$
$$\sum_l a_l^{j,i_k} (x_{i_1} \ldots x_l \ldots x_{i_m})^{\mathrm{sym}} = \sum_l a_l^{j,i_k} \frac{1}{m!} \sum_{\sigma \in S(m)} x_{i_1} \otimes \cdots \otimes \underbrace{x_l}_{\sigma(i_k)\text{th spot}} \otimes \cdots x_{\sigma(i_m)}.$$

We switch the sums in l and σ in the above expression. For every σ, we find x_l in a fixed spot of the tensor product, and the above is equal to

$$\frac{1}{m!} \sum_{\sigma \in S(m)} \sum_l x_{\sigma(i_1)} \otimes \cdots \otimes \left(a_l^{j,i_k} x_l \right) \otimes \cdots \otimes x_{\sigma(i_m)} = \frac{1}{m!} \sum_{\sigma \in S(m)} x_{\sigma(i_1)} \otimes \cdots \otimes \overbrace{[x_j, x_{i_k}]}^{\sigma(i_k)\text{th spot}} \otimes \cdots \otimes x_{\sigma(i_m)}.$$

The sum of these terms over all k and all σ in (5) equals the left hand side (4), hence we get our claim (3).

4. Show that given abelian Lie algebras $\mathfrak{h}$ and $\mathfrak{h}'$, there is a one-to-one correspondence between Lie algebras $\mathfrak{g}$ such that

$$\mathfrak{z}(\mathfrak{g}) \cong \mathfrak{h} \qquad \text{and} \qquad \mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \cong \mathfrak{h}'$$

(such $\mathfrak{g}$ are called **2-step nilpotent**) and non-degenerate anti-symmetric bilinear forms

$$\mathfrak{h}' \times \mathfrak{h}' \to \mathfrak{h}$$

Solution. Let us consider a Lie algebra $\mathfrak{g}$ such that $\mathfrak{z}(\mathfrak{g}) \cong \mathfrak{h}$ and $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \cong \mathfrak{h}'$. For every $x \in \mathfrak{g}$, denote as $\overline{x}$ the image in $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$. Then,

$$\overline{[x,y]} = [\overline{x}, \overline{y}] = 0$$

for all $x, y \in \mathfrak{g}$, since $\mathfrak{h}'$ is abelian. This implies that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}(\mathfrak{g})$, and the Lie bracket $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ induces to a bilinear, non-degenerate, anti-symmetric form $\mathfrak{h}' \times \mathfrak{h}' \to \mathfrak{h}$. Viceversa, let $\phi\colon \mathfrak{h}' \times \mathfrak{h}' \to \mathfrak{h}$ be a non-degenerate anti-symmetric bilinear form. Then endow the vector space $\mathfrak{g} = \mathfrak{h}' \oplus \mathfrak{h}$ with the Lie bracket

$$[(x_1, y_1), (x_2, y_2)] = (0, \phi(x_1, x_2)).$$

It is clear that $\mathfrak{h} \subset \mathfrak{z}(\mathfrak{g})$, and the other inclusion follows from the fact that ϕ is non-degenerate. Finally, the isomorphism $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \cong \mathfrak{h}'$ is obvious.

5. Consider any matrix Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_n$ (for some n large enough). Show that

$$\begin{aligned}
\mathfrak{h} &= \mathfrak{g} \cap \Big\{\text{diagonal } n \times n \text{ matrices}\Big\}\\
\mathfrak{n} &= \mathfrak{g} \cap \Big\{\text{strictly upper triangular } n \times n \text{ matrices}\Big\}\\
\mathfrak{b} &= \mathfrak{g} \cap \Big\{\text{upper triangular } n \times n \text{ matrices}\Big\}
\end{aligned}$$

are abelian, nilpotent, solvable Lie subalgebras of $\mathfrak{g}$ (respectively).

Solution. By Proposition 11 in the Lecture notes, we have to verify that

$$\begin{aligned}
\mathfrak{h}_n &= \Big\{\text{diagonal } n \times n \text{ matrices}\Big\}\\
\mathfrak{n}_n &= \Big\{\text{strictly upper triangular } n \times n \text{ matrices}\Big\}\\
\mathfrak{b}_n &= \Big\{\text{upper triangular } n \times n \text{ matrices}\Big\}
\end{aligned}$$

are abelian, nilpotent, solvable Lie subalgebras of $\mathfrak{gl}_n$ respectively. The claim is trivial for $\mathfrak{h}_n$. Let us consider $\mathfrak{n}$, whose elements are strictly upper triangular matrices. Then, very matrix in $[\mathfrak{n}_n, \mathfrak{n}_n]$ has zeros on the main diagonal and on the secondary diagonal

$$\begin{bmatrix}
0 & 0 & * & \dots & * \\
0 & 0 & 0 & *\dots & * \\
\vdots & \dots & \ddots & \ddots & \vdots \\
0 & 0 & \dots & \dots & 0 \\
0 & 0 & \dots & \dots & 0 \\
.
\end{bmatrix}$$

Iterating, this shows that eventually the series $\mathfrak{n}_n \supseteq [\mathfrak{n}_n, \mathfrak{n}_n] \supseteq \dots$ is zero. Finally, we observe that

$$[\mathfrak{b}_n, \mathfrak{b}_n] \subset \mathfrak{n}_n.$$

Then, the fact thet $\mathfrak{n}_n$ is nilpotent implies that $\mathfrak{b}_n$ is solvable.