MATH 429 EXAM - 16/06/2025 (180 MINUTES)

No books, notes, or electronic devices (especially no phones) are permitted during this exam.

You must show your work to receive credit. **Justify everything**.

Do not unstaple the exam or reorder the pages. All problems must be solved within the space provided (right after the statement of the problem). If you need to use the extra pages at the end, then mention this clearly in the aforementioned space, so your grader knows that they have to also look at the end (they will not check the extra pages unless explicitly told to).

We will provide scratch paper (loose sheets) but do not write solutions on them. **Only the 16 pages of the booklet you're now reading will be graded**.

Please do not leave the room during the first and last 30 minutes of the exam.

Please keep your CAMIPRO face up on the table at all times.

Don't forget to write your name, SCIPER, and sign the exam.

There are 7 problems, worth 100 points in total (note that Problem 7 has a part (b)).

NAME: ______________________________

SCIPER: ______________________________

SIGNATURE: ______________________________

PROBLEM 1

Consider the 3-dimensional Lie algebra $\mathfrak{g} = \mathbb{C}a \oplus \mathbb{C}b \oplus \mathbb{C}c$ with Lie bracket defined by

$$[a, b] = b, \qquad [a, c] = c, \qquad [b, c] = 0$$

Show that this Lie algebra is solvable, in two ways:

(a) by considering its derived series $\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \supseteq \dots$ *(8 points)*

(b) by verifying Cartan's criterion $\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_x \mathrm{ad}_y) = 0$ for all $x \in \mathfrak{g}$, $y \in [\mathfrak{g}, \mathfrak{g}]$. *(7 points)*

PROBLEM 2

Consider the irreducible representation $\mathfrak{sl}_2 \curvearrowright L(n)$ with highest weight $n \in \mathbb{Z}_{\geq 0}$, and the Verma module (below, $\mathfrak{b}$ denotes the subalgebra of $\mathfrak{sl}_2$ spanned by E and H)

$$M(n) = U\mathfrak{sl}_2 \bigotimes_{U\mathfrak{b}} \mathbb{C}$$

generated by a single vector $v = 1 \otimes 1$ satisfying the relations $E \cdot v = 0$ and $H \cdot v = nv$.

(a) Construct an explicit basis of $M(n)$ as an infinite-dimensional vector space, and prove explicit formulas for the action of the operators $E, F, H \in \mathfrak{sl}_2$ in this basis. *(8 points)*

(b) Identify the kernel $K(n)$ of the natural surjection $M(n) \twoheadrightarrow L(n)$. *(5 points)*

(More specifically, you should prove that $K(n)$ *is isomorphic to some representation of* $\mathfrak{sl}_2$ *which we have "named" in class, such as an irreducible representation or a Verma module)*

(c) Recall that the character of a representation $\mathfrak{sl}_2 \curvearrowright V$ is the formal sum

$$\chi_V = \sum_{\ell \in \mathbb{C}} t^\ell \cdot \dim_{\mathbb{C}} \Big(\text{weight } \ell \text{ subspace of } V \Big)$$

(note that χ_V may be a power series if V is infinite dimensional). Calculate $\chi_{L(n)}$, $\chi_{M(n)}$ and $\chi_{K(n)}$, where $K(n)$ denotes the kernel from part (b). *(6 points)*

PROBLEM 3

(a) We know that $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$ are complex Lie groups with Lie algebras $\mathfrak{sl}_n$, $\mathfrak{o}_n$, $\mathfrak{sp}_{2n}$, respectively. The latter are simple Lie algebras, so therefore

$$\mathfrak{g} = \mathfrak{sl}_{20} \oplus \mathfrak{o}_{25}$$

is a semisimple Lie algebra. Construct a complex Lie group with Lie algebra $\mathfrak{g}$. *(6 points)*

(b) Give an example of two non-isomorphic Lie groups (either real or complex) with isomorphic Lie algebras. Justify your assertions. *(6 points)*

PROBLEM 4

Let G be the Lie group of invertible functions $\mathbb{R} \xrightarrow{f_{a,b}} \mathbb{R}$, $f_{a,b}(x) = ax + b$. In other words

$$G = \mathbb{R}^* \times \mathbb{R}, \qquad f_{a,b} \rightsquigarrow (a, b)$$

with the group operation induced by composition of functions. Write out this operation explicitly by filling in the blanks below *(7 points)*

$$(a, b) \cdot (a', b') = \Big(\underline{\qquad\qquad} , \underline{\qquad\qquad} \Big)$$

Determine $\mathfrak{g} = \mathrm{Lie}(G)$ and its Lie bracket. *(5 points)*

(You may express elements of G near the identity $1 \in G$ as $g = 1 + \varepsilon y$ where $y \in \mathfrak{g}$ and ε is infinitesimally small. Then by considering $gg'g^{-1} \in G$ for group elements $g = 1 + \varepsilon y$ and $g' = 1 + \varepsilon' y'$, the order $\varepsilon\varepsilon'$ term recovers the Lie bracket $[y, y'] \in \mathfrak{g}$)

PROBLEM 5

(a) What do we mean when we say that a (complex, finite-dimensional) representation V of a (complex, finite-dimensional) semisimple Lie algebra is completely reducible?

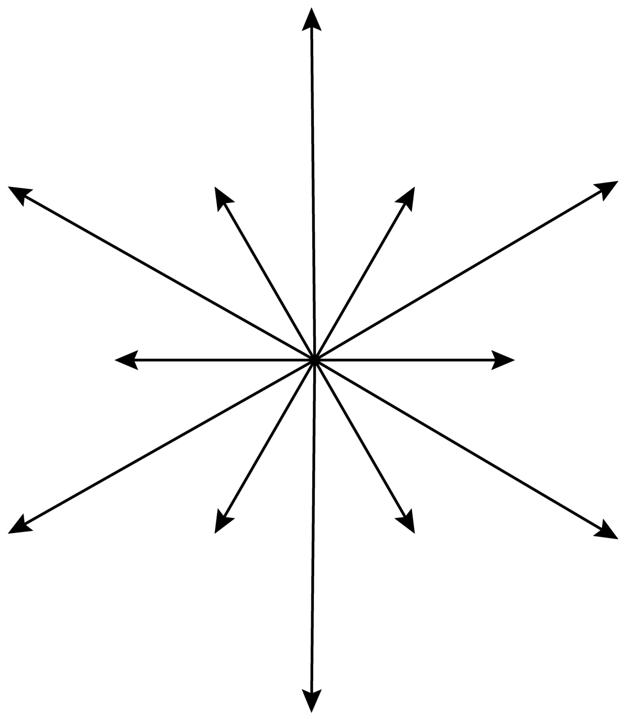
(we encountered several, essentially equivalent, formulations of complete reducibility; stating any one of them in the space above would be acceptable) *(7 points)*

(b) A (complex, finite-dimensional) Lie algebra $\mathfrak{g}$ is called reductive if $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is semisimple. Prove that the adjoint representation of a reductive Lie algebra $\mathfrak{g}$ is completely reducible.

(5 points)

PROBLEM 6

The following is a picture of the root system of type G_2 (the long vectors are $\sqrt{3}$ times bigger than the short vectors, and the angles between adjacent vectors are all equal).



(a) Draw any half-plane whose boundary line passes through the origin but does not contain any of the root vectors: the 6 positive roots are those ones which lie in the chosen half plane.

Indicate in the picture the 2 simple roots α_1 and α_2 corresponding to your choice of half-plane.

Write on the picture the other 4 positive roots as explicit linear combinations of α_1 and α_2.

(9 points)

(b) Determine, with proof, the Weyl group corresponding to the root system of type G_2.

(6 points)

PROBLEM 7

For any semisimple (complex, finite-dimensional) Lie algebra $\mathfrak{g}$ with root decomposition

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

consider the subspace

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$$

defined with respect to some decomposition $R = R^+ \sqcup R^-$ into positive and negative roots.

(a) Show that $\mathfrak{n}^+$ is a Lie subalgebra of $\mathfrak{g}$. *(9 points)*

(b) Show that $\mathfrak{n}^+$ is a nilpotent Lie algebra. *(6 points)*

EXTRA SHEET

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