This problem set is not representative of this class. We just want to collect some useful tools from functional analysis. Then we will move on!

Exercise 0

Make sure that you know the following notions, using, if needed, the library and Wikipedia:

(i) the dual V^* of a normed vector space V;
(ii) the weak topology on V and the weak-* topology on V^*;
(iii) the norm of a linear map between normed vector spaces;
(iv) the Hahn–Banach theorem for normed vector spaces.

Exercise 1

Let V be a Banach space. Define $\iota\colon V \to V^{**}$ by $\iota(v)(f) = f(v)$. Prove:

(i) ι is well-defined, linear and isometric.

(ii) ι is a homeomorphism onto its image when V has the weak topology and V^{**} the weak-* topology.

(iii) $\iota(V)$ is weak-* dense in V^{**}.

(iv) If $\lambda \in V^{**}$ is weak-* continuous as a map $V^* \to \mathbf{R}$, then $\lambda \in \iota(V)$.

Exercise 2

Let V, W be Banach spaces.

(i) Given a continuous linear map $\alpha\colon V \to W$, define $\alpha^*\colon W^* \to V^*$ and check that $\|\alpha^*\| = \|\alpha\|$.

(ii) Let $\beta\colon W^* \to V^*$ be a continuous linear map. Prove that $\beta = \alpha^*$ for some $\alpha\colon V \to W$ if and only if β is continuous for the weak-* topologies. *Hint: use Ex. 1(iv).*

Exercise 3

Let X be a set and define $\ell^1(X) = \big\{f\colon X \to \mathbf{R} : \|f\|_1 < \infty\big\}$, where $\|f\|_1 = \sum_{x\in X} |f(x)|$.

(i) Make sure that you really understand the definition of $\sum_{x\in X} |f(x)|$ and deduce that $\sum_{x\in X} f(x)$ makes sense for $f \in \ell^1(X)$. Give a formal definition of $\sum_{x\in X} f(x)$.

Next, define $\ell^\infty(X) = \big\{f\colon X \to \mathbf{R} : \|f\|_\infty < \infty\big\}$, where $\|f\|_\infty = \sup_{x\in X} |f(x)|$.

(ii) Given $a \in \ell^1(X)$ and $b \in \ell^\infty(X)$, we can define the number $\langle a, b\rangle = \sum_{x\in X} a(x)b(x)$. (BTW: why?) Use this to construct an isometric isomorphism between $\ell^\infty(X)$ and $\ell^1(X)^*$.

Additional question: why does it not *similarily identify $\ell^1(X)$ with $\ell^\infty(X)^*$?*