Exercise 0 (reminders about isometries). The group G of isometries of $\mathbf{R}^n$ consists, by definition, of all bijections preserving the Euclidean distance. Prove $G = \mathbf{R}^n \rtimes \mathbf{O}(n)$.

Hints:

(i) Verify that translations and elements of $\mathbf{O}(n)$ are isometries, and form a semi-direct product $\mathbf{R}^n \rtimes \mathbf{O}(n)$.

(ii) Show that any isometry g fixing 0 must satisfy

$$g\left(\frac{x+y}{2}\right) = \frac{g(x)+g(y)}{2}$$

for all $x, y \in \mathbf{R}^n$.

(iii) Show that any isometry fixing 0 must be linear.

(iv) Complete the proof.

Exercise 1 (a free group of rotations). Consider the rotations

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}, \qquad b = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{where } c = \frac{1}{3} \text{ and } s = \frac{2\sqrt{2}}{3}.$$

You might want to check that $a, b \in \mathbf{SO}(3)$ and give a geometric description of a and b in everyday language; think "cos" and "sin".

The goal of this exercise is to show that a, b generate a free group F_2. Thus, for a reduced word w of length k in the letters $a^{\pm 1}$, $b^{\pm 1}$, we need to show that w does not represent the trivial rotation. There is no loss of generality in assuming that w ends with $b^{\pm 1}$ (why?). It suffices to prove that $w(1,0,0) \neq (1,0,0)$ — abusively using the horizontal vector notation to save space.

(i) Prove that there are integers $x, y, z \in \mathbf{Z}$ such that $w(1,0,0) = 3^{-k}(x, y\sqrt{2}, z)$.

(ii) Prove that y is not divisible by 3; then in particular $y \neq 0$, finishing the proof of the exercise. You can do this by induction by looking carefully at the first two letters of w and how they change the parameter y.

Exercise 2 (optional; cultural fact about rotations). We denote by SU(2) the group of 2×2 unitary complex matrices with determinant one.

Prove that there is a surjective homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$. What is the kernel?

Hint for finding the homomorphism: try to use complex numbers to describe real vectors.