Exercise 1 (maximum principle). Let G be a group, $\mu \in \mathrm{Prob}(G)$ and suppose that the support of μ generates G as semigroup (or as monoid). Prove the following *maximum principle*:

If a μ-harmonic function f has a maximum on G, then f is constant.

What happens if we assume nothing on the support? or if we only assume that the support of μ generates G as a group? (For these two additional questions, $\mathbf{Z}$ is good enough...)

Exercise 2. The action of a group G on a set X is called **paradoxical** if there is a partition

$$X = A_1 \sqcup \ldots \sqcup A_n \sqcup B_1 \sqcup \ldots \sqcup B_m$$

and group elements $g_1, \ldots, g_n, h_1, \ldots, h_m$ which lead to two new partitions

$$X = g_1 A_1 \sqcup \ldots \sqcup g_n A_n \quad \text{and} \quad X = h_1 B_1 \sqcup \ldots \sqcup h_m B_m.$$

(i) Check that this is impossible if $n + m < 4$.
(ii) Suppose that $n + m = 4$; prove that G contains a subgroup isomorphic to F_2.

*Note: the smallest possible integer $n + m$ is called the **Tarski number** of the action $G \curvearrowright X$.*

Exercise 3. Let G be a group acting on a set X and let $A, B \subseteq X$.

(i) Suppose that there is a piecewise-G surjection $A \to B$. Prove that there is a piecewise-G injection $B \to A$.

(ii) Show that the converse does not hold.