Exercise 1. Let G be a group acting on a set X. Show that the **Følner property**

(F) $$\forall\, S \subseteq_{\rm f} G\ \forall\, \epsilon > 0\ \exists\, A \subseteq_{\rm f} X\ \forall\, s \in S : |sA \triangle A| < \epsilon |A|$$

is equivalent to:

(F′) $$\forall\, S \subseteq_{\rm f} G\ \forall\, \epsilon > 0\ \exists\, A \subseteq_{\rm f} X : |SA| < (1+\epsilon)|A|.$$

Note: by definition, $SA = \{sa : s \in S, a \in A\}$.

Exercise 2. A group G is called **sub-exponential** if $\lim_{n\to\infty} \sqrt[n]{|C^n|} = 1$ for every finite subset $C \subseteq_{\rm f} G$, where $C^n = C \cdots C$.

(n) Explain why this limit exists!

(i) Prove that every sub-exponential group is amenable.

(ii) Check that it applies to $G = \mathbf{Z}^d$.

(iii) Optional: consider the group G of maps $\mathbf{R} \to \mathbf{R}$ generated by $a(x) = 2x$ and $b(x) = x + 1$. Prove that G is not sub-exponential. (But you know that G is amenable.)

Hints for (ii): take it easy and start with $d = 1$*, then* $d = 2$*.*
Begin by observing that you can replace C *by another set* C_0 *as long as there is some* n_0 *such that* $C \subseteq C_0^{n_0}$*. So, choose a set* C_0 *which is simple and takes care of all possible* C *at once. . .*

Exercise 3 (a variant of Reiter). Let G be a group acting on a set X. Prove that the following condition is equivalent to (R_2):

For every $S \subseteq_{\rm f} G$ and every $\varepsilon > 0$ there is $v \in \ell^2(X)$ with

$$\left\| \frac{1}{|S|} \sum_{s\in S} sv \right\|_2 > (1-\varepsilon)\|v\|_2.$$

Optional questions:

- Harder: how about the above condition but with $\|\cdot\|_p$ for a general $p > 1$?
- Easier: and how about $p = 1$?