Exercise 0. Let G be a group, $f \in \ell^\infty(G)$ and $h \in \ell^1(G)$. Show that the following are equal for all $x \in G$ (and pay attention to absolute convergence).

$$\sum_{y \in G} f(xy^{-1})\, h(y), \qquad \sum_{y \in G} f(y)\, h(y^{-1}x), \qquad \sum_{\{y,z \in G \,:\, yz = x\}} f(y)\, h(z).$$

Exercise 1. Let f, g, h be functions on a group G and let $x, y \in G$.

(i) Verify that $f*(g*h) = (f*g)*h$ holds (as soon as all these sums are absolutely convergent).

(ii) Compute $\delta_x * \delta_y$, $\delta_x * f$ and $f * \delta_x$.

(iii) Find a formula for $(f * g)^\vee$, where for any function h we write $h^\vee(x) := h(x^{-1})$.

(iv) Let $1 \leq p \leq \infty$ and suppose $f \in \ell^p(G)$, $g \in \ell^1(G)$. Verify $\|g * f\|_p \leq \|f\|_p \cdot \|g\|_1$. Deduce $\|f * g\|_p \leq \|f\|_p \cdot \|g\|_1$. (You might want to separate the case $p = \infty$.)

(v) For $f \in \ell^1(G)$, write $\Sigma f := \sum_{x \in G} f(x)$. Assuming $f, g \in \ell^1(G)$, prove $\Sigma(f*g) = (\Sigma f)(\Sigma g)$. Deduce that $f, g \in \mathrm{Prob}(G)$ implies $f * g \in \mathrm{Prob}(G)$.

Exercise 2. In class, we found a function $f \colon F_2 \to \mathbf{R}$ on $F_2 = \langle a, b \rangle$ that is μ-harmonic for

$$\mu = \frac{1}{4}(\delta_a + \delta_b + \delta_{a^{-1}} + \delta_{b^{-1}}).$$

Find your own μ-harmonic function on F_2 for the same measure μ.
Don't cheat by choosing simply a linear combination of f and $\mathbf{1}_G$...

Exercise 3. On the group $\mathbf{Z}$ with the probability measure $\mu = \frac{1}{3}\delta_{-1} + \frac{2}{3}\delta_1$, determine all μ-harmonic functions.