Exercise 0. In the proof of the Barycenter Theorem, track what happens in every step of the proof if $\mu \in \mathscr{M}(K)$ is a finite convex combination of Dirac masses.

That is, let $x_1, \ldots, x_n \in K$, let $t_i \geq 0$ with $t_1 + \cdots + t_n = 1$ and let $\mu = t_1\delta_{x_1} + \cdots + t_n\delta_{x_n}$. Now make each step of the proof concrete and explicit.

Exercise 1 (On $\mathrm{SL}_2(\mathbf{Z})$, part II). The goal of this exercise is to prove that the group $\mathrm{SL}_2(\mathbf{Z})$ contains a free group F_2.

Consider the elements $u = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $v = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$. Analyse what they do to a vector $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ in $\mathbf{Z}^2$ by discussing whether $|x| < |y|$ etc.

Now prove that the subgroup $\langle u, v \rangle$ generated by $\{u, v\}$ is the free group on $\{u, v\}$.

Hint: first, understand that this amounts simply to proving that every (non-empty) ***reduced*** *word using u, v represents a non-trivial element of* $\mathrm{SL}_2(\mathbf{Z})$.

Exercise 2. Find a sequence G_n of amenable groups such that the product $\prod_n G_n$ is non-amenable.

Hint: Using the result about subgroups of amenable groups, Ex. 1 above implies that the group $\mathrm{SL}_2(\mathbf{Z})$ *is non-amenable. Now, try to take all G_n finite...*

Exercise 3 (On $\mathrm{SL}_2(\mathbf{Z})$, part III). Prove that $\mathrm{SL}_2(\mathbf{Z})$ is "virtually free" in the sense that the quotient $\mathrm{SL}_2(\mathbf{Z})/\langle u, v \rangle$ is finite.

Hint: it may help to find a concrete description of this quotient.