Exercise 1
Let X be a set and $m \in \ell^\infty(X)^*$. Consider the following properties:
(i) $m(f) \geq 0$ if $f \geq 0$.
(ii) $m(\mathbf{1}_X) = 1$, where $\mathbf{1}_X$ is the constant function 1.
(iii) $\|m\| = 1$.
Show that any two of these properties imply that all three hold.

Exercise 2
(i) Prove that the image of the Dirac mass map $\delta\colon X \to \mathscr{M}(X)$ is a *discrete* subspace.

From now on, we identify (abusively) this image with X.
Define βX to be the closure of X in $\mathscr{M}(X)$.

(iii) Give a precise proof that $\beta X \neq X$ when X is infinite.

(iii) Prove $\beta X = \big\{\mu \in \mathscr{M}(X) : \mu(A) \in \{0,1\}\ \forall\, A \subseteq X\big\}$.

(iv, optional) Prove that $\beta\mathbf{N}$ is a separable compact space but is not second countable.

(v) Easier than last week: show that the sequence δ_n has no convergent subsequence in the compact space $\beta\mathbf{N}$.

Exercise 3
Let $f\colon X \to X$ be a bijection of a set X.

(i) Show that there exists a (unique) homeomorphism $\beta f\colon \beta X \to \beta X$ extending f.

(ii) Show that $\beta f\colon \beta X \to \beta X$ has a fixed point (if and) only if f does.

Hint: Show first that the extension to $\beta\mathbf{Z} \to \beta\mathbf{Z}$ of the map $\mathbf{Z} \to \mathbf{Z}$, $n \mapsto n+1$ has no fixed point, then try to generalize.

Moral of the exercise: invariant means in $\mathscr{M}(X)$ are generally not in $\beta X \subseteq \mathscr{M}(X)$.

Exercise 4 (On $\mathrm{SL}_2(\mathbf{Z})$)
Consider the elements $a = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ of the group $\mathrm{SL}_2(\mathbf{Z})$.

(i) Prove that a and b generate $\mathrm{SL}_2(\mathbf{Z})$.

(ii) Prove that a and b satisfy a non-trivial relation.