Exercise 0. Check that the entire proof of the Banach–Tarski theorem holds more generally in $\mathbf{R}^n$ for all $n \geq 3$.
What goes wrong for $n = 2$?

Exercise 1. The Banach–Tarski theorem states that the closed unit ball $B \subseteq \mathbf{R}^3$ is equidecomposable with every subset $A \subseteq \mathbf{R}^3$ that is bounded and of non-empty interior.

(i) Give a precise proof that this fails for every A that is not bounded.

(ii) Give an example to show that this fails for some bounded A with empty interior.

(iii) Give an example to show that this holds for some A with empty interior.

Exercise 2. Let $G = \mathrm{Isom}(\mathbf{R}^4)$. Let $B_3 \subseteq \mathbf{R}^4$ be the closed unit ball of $\mathbf{R}^3$ viewed as a subset of $\mathbf{R}^4 = \mathbf{R}^3 \oplus \mathbf{R}$; thus B_3 has empty interior.

(i) Explain why it is still true that B_3 is G-equidecomposable to two disjoint copies of B_3 in $\mathbf{R}^4$.

(ii) Is B_3 G-equidecomposable to the unit ball B_4 of $\mathbf{R}^4$?

Exercise 3 (On $\mathrm{SL}_2(\mathbf{Z})$, part IV). Consider the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{Z}^2$ (by matrix multiplication) and write 0 for the zero vector in $\mathbf{Z}^2$.

(i) Prove that δ_0 is the only $\mathrm{SL}_2(\mathbf{Z})$-invariant mean on $\mathbf{Z}^2$.

Hint: your solution to Ex. 1 in Problem Set 4 certainly contains the ingredients that you need.

(ii) Consider the corresponding semi-direct product $G = \mathbf{Z}^2 \rtimes \mathrm{SL}_2(\mathbf{Z})$. Deduce from (i) that $\mathrm{SL}_2(\mathbf{Z})$ is not co-amenable in G.