The (integral) **Heisenberg group** G can be defined as the set $\mathbf{Z}^3$ with the following multiplication:

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy').$$

This looks like the usual group structure of $\mathbf{Z}^3$ perturbed by the term xy'; for those who know cohomology, the latter term is a "2-cocycle". Observe that G is not commutative:

$$(1, 0, 0)(0, 1, 0) = (1, 1, 1) \neq (1, 1, 0) = (0, 1, 0)(1, 0, 0).$$

Check that this multiplication is indeed associative. The neutral element is $(0, 0, 0)$*; what is the inverse of* (x, y, z) *?*

We can define a subgroup $Z < G$ by $Z = \{(0, 0, z) : z \in \mathbf{Z}\}$. This is a normal subgroup, because it has the much stronger property of being **central**: the elements of Z commute with every element of G.

Verify this claim, and verify that conversely every such element belongs to Z*. In words:* Z *is the* ***center of*** G.

Consider the map $\pi\colon G \to \mathbf{Z}^2$ defined by $\pi(x, y, z) = (x, y)$. This is a group homomorphism, it is surjective, and its kernel is precisely Z. In conclusion, we have shown that G is an extension of $\mathbf{Z}$ by $\mathbf{Z}^2$.

This gives us a nice example of a non-commutative amenable group. In contrast to semi-direct products, we cannot lift $\mathbf{Z}^2$ into G with respect to π: even though there are subgroups of G isomorphic to $\mathbf{Z}^2$ (do you see them?), such subgroups will never map onto $\mathbf{Z}^2$ via π. (Do you see why not?). For instance, the elements of the form $(x, y, 0)$ do not form a subgroup.

...if anyone likes matrices, they can also think of G in terms of $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$.