Answer sheet 6

Assignment 1. (a) The Neyman–Pearson most powerful test is given by the rejection region

$$\frac{(\sqrt{2\pi}\sigma_1)^{-n}\exp\{-\sum_{i=1}^n X_i^2/(2\sigma_1^2)\}}{(\sqrt{2\pi}\sigma_0)^{-n}\exp\{-\sum_{i=1}^n X_i^2/(2\sigma_0^2)\}} > k$$

which is equivalent to

$$\sum_{i=1}^n X_i^2 < c$$

since $\sigma_1^2 < \sigma_0^2$.
To determine the critical value c it is enough to realize that the $\sum_{i=1}^n X_i^2/\sigma_0^2 \sim \chi_n^2$ under H_0. Denote the α-quantile of the χ_n^2 distribution by c_α, i.e., $c_\alpha = H_n^{-1}(\alpha)$, where H_n is the cdf of the χ_n^2 distribution. Therefore, the critical value is $c = \sigma_0^2 c_\alpha$.
No, the critical value of the test does not depend on σ_1^2.
(b) The power against $\sigma_1^2 < \sigma_0^2$ equals

$$P_{\sigma_1^2}\left(\frac{\sum_{i=1}^n X_i^2}{\sigma_0^2} < c_\alpha\right) = P_{\sigma_1^2}\left(\frac{\sum_{i=1}^n X_i^2}{\sigma_1^2} < c_\alpha\frac{\sigma_0^2}{\sigma_1^2}\right) = H_n\left(c_\alpha\frac{\sigma_0^2}{\sigma_1^2}\right).$$

(c) The minimal sample size needed to reject H_0 with probability β when the true variance is σ_1^2 is given implicitly as the solution to

$$H_n\left(c_\alpha\frac{\sigma_0^2}{\sigma_1^2}\right) \geq \beta.$$

Assignment 2. (a) The Neyman–Pearson test rejects for

$$\frac{p_1^T(1-p_1)^{n-T}}{p_0^T(1-p_0)^{n-T}} > k,$$

where $T = \sum_{i=1}^n X_i$. Equivalently, it rejects for

$$\left[\frac{p_1(1-p_0)}{p_0(1-p_1)}\right]^T\left[\frac{1-p_1}{1-p_0}\right]^n > k.$$

Since $\frac{p_1(1-p_0)}{p_0(1-p_1)} > 1$ (because $p_1 > p_0$), the critical region can be further simplified to $T > c$. Under H_0, the statistic $T = \sum_{i=1}^n X_i \sim Bin(n, p_0)$. Let $c = c_{1-\alpha}$ be the $(1-\alpha)$-quantile of this distribution, i.e., $c_{1-\alpha} = \inf\{x : G_{n,p_0}(x) \geq 1-\alpha\} = G_{n,p_0}^{-}(1-\alpha)$, where G_{n,p_0} denotes the cdf of the $Bin(n, p_0)$ distribution. If $1 - G_{n,p_0}(c_{1-\alpha}) = \alpha$, then the test $T > c_{1-\alpha}$ is the most powerful test of significance level α of the test. Otherwise, when

$$\mathbb{P}_{H_0}(T > c_{1-\alpha}) < \alpha < \mathbb{P}_{H_0}(T \geq c_{1-\alpha}),$$

we do not get a most powerful test.
No, when a most powerful test exists, the critical value of the test does not depend on p_1.
(b) For $p_0 = \frac{3}{10}$, $n = 3$, we have $P(T=3) = \binom{3}{3}(\frac{3}{10})^3 = \frac{27}{1000}$, $P(T=2) = \binom{3}{2}(\frac{3}{10})^2(1-\frac{3}{10}) = \frac{189}{1000}$. Thus, rejecting for $T > 2$ would give level 0.027 while rejecting for $T > 1$ would give

level $0.027+0.189 = 0.216$. Therefore, there does not exist a most powerful test at significance level $\alpha = 0.05$.
(c) However, there exists a most powerful test at the significance level $\alpha = 0.027$. It is given by $\delta(X_1, X_2, \ldots, X_n) = \mathbf{1}(T > 2)$.
(d) The statistic

$$\frac{T - np_0}{\sqrt{np_0(1-p_0)}}$$

is asymptotically standard normal under H_0. Hence, the asymptotic significance level-α test rejects when this statistic exceeds the $(1-\alpha)$-quantile of the $N(0,1)$ distribution.

Assignment 3. (a) The likelihood ratio is

$$\Lambda(X_1, \ldots, X_n) = \frac{5^n e^{-5\sum X_i}}{4^n e^{-4\sum X_i}} = \exp(n\log(5/4) - \sum X_i).$$

We reject H_0 if $\Lambda(X_1, \ldots, X_n)$ is large ; equivalently, if $T(X_1, \ldots, X_n) = \sum X_i$ is small. The test function is therefore $\delta(X_1, \ldots, X_n) = 1$ if $T \leq q$ and 0 otherwise, where q is such that $\mathbb{P}(\sum X_i \leq q) = \alpha$.
(b) The moment generating function of the sum is

$$\prod_{i=1}^{n} \frac{\lambda}{\lambda - t} = \left(\frac{\lambda}{\lambda - t}\right)^n$$

which is the moment generating function of a $Gamma(n, \lambda)$ random variable.
(c) The test function is the same for all $\lambda_1 > 4$. This test can be shown to be uniformly optimal. (If $\lambda_1 < 4$, we would reject when $\sum X_i$ is large.)
(d) Under H_0, $T \sim Gamma(n, 4)$. Therefore q is the α-quantile of the $Gamma(n, 4)$ distribution. This is a continuous distribution, so q exists, and it is unique because the density is positive on $[0, \infty)$.
(e)–(g) We may use the following code :

```
set.seed(18102017)
lambda <- 4
n <- 17
REP <- 1000
alpha <- 0.05
rej <- logical(REP)
q <- qgamma(alpha, shape = n, rate = 4)
for(i in 1:REP)
{
  X <- rexp(n, rate = lambda)
  rej[i] <- (sum(X) <= q)  #### returns 1 if the condition is satisfied, 0 otherwise
}
mean(rej)
```

(h) When $\lambda = 4$, we indeed reject approximately 50 times, namely 5%. When $\lambda = 3$ we reject less ; the test is conservative and the type I error is smaller than 5%. When $\lambda = 5$ we reject more (209 times in this particular example), so the power is approximately 0.209 ; as λ

becomes larger we reject more and more and the power increases and approaches one. These two phenomena (increase of power and decrease of type I error) occur more rapidly the larger n is.

Assignment 4. (i) The likelihood function for the sample is the joint probability function of all x_i's and y_i's and is given by

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{\theta_1^{x_i} e^{-\theta_1}}{x_i!} \prod_{i=1}^{n} \frac{\theta_2^{y_i} e^{-\theta_2}}{y_i!} = \left(\frac{1}{k}\right) \theta_1^{\sum_{i=1}^n x_i} e^{-n\theta_1} \theta_2^{\sum_{i=1}^n x_i} e^{-n\theta_2}$$

where $k = x_1! \dots x_n! y_1! \dots y_n!$ and $n = 100$.
We can see that $L(\theta_1, \theta_2)$ is maximised when both θ_1 and θ_2 are equal to their m.l.e. $\theta_1 = \bar{x}$ and $\theta_2 = \bar{y}$.
Moreover under H_0 the likelihood is

$$L(\theta) = \left(\frac{1}{k}\right) \theta^{\sum_{i=1}^n x_i + \sum_{i=1}^n y_i} e^{-2n\theta},$$

a function of only one parameter $\theta = \theta_1 = \theta_2$ maximised in

$$\hat{\theta} = \frac{1}{2n} \left(\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i \right) = \frac{1}{2}(\bar{x} + \bar{y}).$$

In this example the parameter space is $\Theta = \{(\theta_1, \theta_2) : \theta_1 > 0, \theta_2 > 0\}$, and we can write the likelihood ratio as

$$\Lambda = \frac{L(\hat{\theta})}{L(\theta_0)} = \frac{\bar{x}^{n\bar{x}} \bar{y}^{n\bar{y}}}{\hat{\theta}^{n\bar{x}+n\bar{y}}}.$$

(ii) We will actually need only the value of $\log \Lambda$, which is easier to compute $\log(\Lambda) = 4.76$. Note that straightforward evaluation of Λ e.g. in R results in NaN.
(iii) $2\log\Lambda$ is an approximate χ_1^2 distribution, therefore we would reject the null hyphothesis for value of $2\log\Lambda$ larger than the $k = 3.841$, where k is such that $\mathbb{P}_0[\Lambda \geq k] = \alpha$. In our case $2\log\Lambda = 9.52$ hence we reject the null hypothesis $\theta_1 = \theta_2$.

Assignment 5. (a) We know that $\sqrt{n}(\widehat{\theta}_n - \theta) \to N(0, 1/I_1(\theta))$. The asymptotic variance is therefore $v(\theta) = 1/(nI_1(\theta))$.
(b) We have

$$T = nI_1(\widehat{\theta})(\widehat{\theta} - \theta_0)^2.$$

(c) Since v is continuous $v(\widehat{\theta})/v(\theta) \to 1$ in probability. By Slutsky's theorem

$$T = nI_1(\theta)(\widehat{\theta} - \theta_0)^2 \frac{v(\theta)}{v(\widehat{\theta})} = \left(\sqrt{nI_1(\theta)}(\widehat{\theta} - \theta_0)\right)^2 \frac{v(\theta)}{v(\widehat{\theta})} \to \chi_1^2.$$

(d) Write $\theta = \sigma^2$ to avoid differentiation errors. The log likelihood and its derivatives are

$$\ell(x_1, \dots, x_n; \theta) = -\frac{n}{2}\ln(2\pi\theta) - \frac{\sum_{i=1}^n x_i^2}{2\theta}$$

$$\ell'(x_1, \dots, x_n; \theta) = \frac{\sum x_i^2}{2\theta^2} - \frac{n}{2\theta} \implies \widehat{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i^2.$$

$$\ell''(x_1, \dots, x_n; \theta) = \frac{n}{2\theta^2} - \frac{\sum x_i^2}{\theta^3} \implies \ell''(\widehat{\theta}) = -\frac{n}{2\widehat{\theta}^2} < 0,$$

so $\widehat{\theta}$ is a maximizer and $nI_1(\widehat{\theta}) = I_n(\widehat{\theta}) = -\mathbb{E}\ell''(\widehat{\theta}) = n/2\widehat{\theta}^2$. We obtain the Wald test statistic

$$T = \frac{n}{2\widehat{\theta}^2}(\widehat{\theta} - \theta_0)^2 = \frac{n}{2}\left(1 - \frac{\sigma_0^2}{\widehat{\sigma}^2}\right)^2, \qquad \widehat{\sigma}^2 = \widehat{\theta}.$$

Since T is asymptotically χ_1^2, the approximate Wald test rejects H_0 if T is larger than the $(1-\alpha)$-quantile of the χ_1^2 distribution, $\chi^2_{1,1-\alpha}$.
Remark. The distribution of $\sum x_i^2/\sigma_0^2$ is χ_n^2, so we can get an exact Wald test, but it will not have an explicit form.
(e) The likelihood ratio is

$$\Lambda = \left(\frac{\sigma_0^2}{\widehat{\sigma}^2}\right)^{n/2} \exp\left(\frac{n}{2}\frac{\widehat{\sigma}^2}{\sigma_0^2}\right) \exp\left(-\frac{n}{2}\right).$$

and twice its logarithm is asymptotically χ_1^2. The asymptotic test rejects therefore when

$$n\left[\frac{\widehat{\sigma}^2}{\sigma_0^2} - \log\frac{\widehat{\sigma}^2}{\sigma_0^2} - 1\right] > \chi^2_{1,1-\alpha}.$$

The tests are not the same, but can be shown (by a Taylor expansion, essentially) to be rather close to each other.

Assignment 6. (a) The model

$X_i \sim \mathrm{N}(\mu, \sigma^2)$	for every $i \in \{1, \dots, 12\}$,
$X_1, \dots, X_{12}$	independent,
the parameters μ and σ^2	unknown.

The null and the alternative hypothesis are :

$$H_0 : \mu = 12.2, \quad H_1 : \mu \neq 12.2.$$

(b) As seen in class, you can pick the statistics

$$T = \sqrt{n}\,\frac{\bar{X}_n - \mu_0}{S_n},$$

where μ_0 is the value under H_0, here $\mu_0 = 12.2$.
We can see that T is “small” if H_0 is true, and “large” is H_1 is true. We note as well that $\bar{X}_n$ is an estimator of the true value of μ. So if H_0 is true we expect that $\bar{X}_n \approx \mu_0$ and $T \approx 0$. On the other hand if H_1 is true we expect that $\bar{X}_n \approx \mu \neq \mu_0$ and $T >> 0$ or $T << 0$.
We could also consider $|T|$ as a test statistics and expect small values under H_0 and large under H_1.

(c) Extreme values correspond to a very large $|T|$, that is for $|T| > c$, where c is a critical value.
To find c remember that we want the probability of the type I error (reject H_0 when it’s true) to be equal to α. In our case

$$\alpha = \mathbb{P}_{H_0}(\{T < -c\} \cup \{T > c\}) = 1 - \mathbb{P}_{\mu=\mu_0}(-c \leq T \leq c). \tag{1}$$

We know that

$$\sqrt{n}\,\frac{\bar{X}_n - \mu}{S_n} \sim t_{n-1}.$$

If H_0 is true, $\mu = \mu_0$, so $T \sim t_{n-1}$. Hence to satisfy the condition of (1) we can take $c = t_{n-1}(1 - \alpha/2)$.

(d) $\alpha = 0.05$, so we reject H_0 in favour of H_1 if

$$\left|\sqrt{12}\,\frac{\bar{X}_n - 12.2}{S_n}\right| > t_{11}(0.975).$$

$$\sqrt{12}\,\frac{\bar{X}_n - 12.2}{S_n} = 2.002 \quad \text{and } t_{11}(0.975) = 2.20,$$

and we do not have enough evidence to reject H_0.
(Which doesn't mean that we "accept" H_0 !).

(e) For $\alpha = 0.10$ we reject H_0 in favour of H_1 if

$$\left|\sqrt{12}\,\frac{\bar{X}_n - 12.2}{S_n}\right| > t_{11}(0.95).$$

$$\sqrt{12}\,\frac{\bar{X}_n - 12.2}{S_n} = 2.002 \quad \text{and } t_{11}(0.975) = 1.80,$$

and this time we do reject H_0.
The difference w.r.t. part (d) is that if we allow a bigger type I error we are satisfied with less evidence to make a decision against H_0.

(f)

$$p_{obs} = \mathbb{P}_{H_0}(\{T < -2.002\} \cup \{T > 2.002\}) = 1 - \mathbb{P}_{\mu=\mu_0}(-2.002 \leq T \leq 2.002).$$

If H_0 is true $T \sim t_{11}$, so

$$p_{obs} = 1 - (F_{t_{11}}(2.002) - F_{t_{11}}(-2.002)),$$

where $F_{t_{11}}$ is the cdf of the t_{11} law. Exploiting the symmetry of this distribution around 0 we obtain that

$$p_{obs} = 2\,(1 - F_{t_{11}}(2.002)) = 2(1 - 0.9647) = 0.071.$$

(g) $p_{obs} > 0.05$, so we do not reject H_0 in favour of H_1 at a 5 % level, while $p_{obs} < 0.10$, thus we do reject H_0 at a 10 % significance level.
We could say that p_{obs} is the smallest level for which we would rejectH_0 in favour of H_1.