ASSIGNMENT SHEET 4

Spring 2025

Assignment 1. Let $X_1, \dots, X_n$ be an i.i.d. sample from a probability distribution F. Suppose that $\mathbb{E}[X_1^2] < \infty$. Define $\mu = \mathbb{E}[X_1]$ and $\sigma^2 = \mathrm{Var}(X_1)$. Let $\overline{X}$ be the sample mean and $S^2 = (n-1)^{-1}\sum_{i=1}^n (X_i - \overline{X})^2$ be the sample variance.
(a) Show that $S^2 \xrightarrow{p} \sigma^2$ as $n \to \infty$.
(*Hint : Write* $S^2 = (n-1)^{-1}\sum_{i=1}^n X_i^2 - [n/(n-1)]\overline{X}^2$ *and apply the large of large numbers to the two terms followed by continuous mapping theorem and Slutsky's theorem.*)
(b) Using continuous mapping theorem and Slutsky's theorem, show that

$$T_n := \frac{\sqrt{n}(\overline{X} - \mu)}{S} \xrightarrow{d} N(0, 1)$$

as $n \to \infty$.
(c) Suppose that F is the $N(\mu, \sigma^2)$ distribution. What do you know about the exact sampling distribution of T_n for any fixed $n \geq 2$?
(d) Use part (b) to determine the behaviour as $n \to \infty$ of the exact distribution obtained in part (c). (Optional : Compare your answer with that obtained for Exercise 6 in Week 3.)

Assignment 2. Let $X_1, \dots, X_n \overset{iid}{\sim} Ber(p)$ for some $p \in (0, 1)$. Let $U_n = \overline{X}(1 - \overline{X})$, where $\overline{X}$ is the sample mean.
(a) What is U_n estimating ? Why ?
(b) Is U_n an unbiased estimator of $p(1-p)$? Justify.
(c) Is U_n a consistent estimator of $p(1-p)$? Justify.
(d) Find out the asymptotic distribution of $\sqrt{n}[U_n - p(1-p)]$ as $n \to \infty$.
(*Hint : Use the central limit theorem and the delta method.*)

Assignment 3. Let $X_1, \dots, X_n \overset{iid}{\sim} Ber(p)$ for some $p \in (0, 1)$. Let $V_n = \sum_{i=1}^n X_i$.
(a) Is X_1 unbiased for p ?
(b) What is a minimal sufficient statistic for p ?
(c) Find $W_n = \mathbb{E}[X_1 \mid V_n]$.
(d) Verify that $\mathbb{E}[W_n] = p$. (e) Show directly that $\mathrm{Var}(W_n) \leq \mathrm{Var}(X_1)$. Is equality attained ? (Note : This is a verification of the Rao-Blackwell theorem and W_n is the "Rao-Blackwellised" version of X_1.)
(f) Find the Cramer-Rao lower bound for the variance of an unbiased estimator of p. Is this lower bound attained by any estimator of p ?

Assignment 4. In a casino one plays the following game. You pay 1 franc. With probability $p = 0.49$, you win 2 francs, whereas with probability $1 - p = 0.51$ you do not win anything.
(a) Which random variable X will you use in order to describe this game ?
(b) Suppose you start with 1000 francs and play this game 1000 times. Write a formula for the probability that you have at least 1000 francs at the end.
(c) Use the central limit theorem to approximate the probability in (b). (Your result should depend on the c.d.f. of standard gaussian. You can use the R function `pnorm` to obtain a real number.)
(d) (Optional) Use the R commands `pnorm` and `pbinom` to visualise the approximation in (c).

Assignment 5. Maximising the likelihood is a way to obtain parameter estimators. In this exercise you are asked to compute the m.l.e.'s for the following distributions :

(i) The Bernoulli distribution.
(ii) The Exponential distribution.
(iii) The Normal distribution (for both μ and σ^2)
(iv) The uniform distribution $U[0, \theta]$.

Assignment 6. Maximum likelihood estimation is a recipe to construct estimators. In this assignment we introduce another such recipe, *the method of moments*. Let $X \sim f(x; \theta)$ be a random variable whose distribution depends on a parameter θ. The expectation $\mathbb{E}\, X$ will therefore also depend on θ. (We assume that it is defined for all θ.) Call this function $m(\theta)$.
(a) Let $X_1, \ldots, X_n$ be an independent sample from X. What can you say about $\overline{X}_n = \sum_{i=1}^n X_i/n$ and $m(\theta)$ when n is large ?
(b) Assume that m is continuously invertible. Explain why $\widetilde{\theta} = m^{-1}(\overline{X}_n)$ is a sensible estimator of θ. It is called the *method of moments* estimator of θ.
(c) Suppose that $X \sim Exp(\lambda)$. Find the method of moments estimator $\widetilde{\lambda}$ of λ and compare with the maximum likelihood estimator $\widehat{\lambda}$.
(d) Suppose that $X \sim Unif(0, \kappa)$. Find the method of moments estimator $\widetilde{\kappa}$ of κ and compare with the maximum likelihood estimator $\widehat{\kappa}$.
(e) Suppose that $f(x; \theta) = \theta x^{-\theta-1}$ for $x \geq 1$ and zero otherwise (as in a previous assignment), with $\theta > 1$. Find the method of moments estimator $\widetilde{\theta}$ of θ and compare with the maximum likelihood estimator $\widehat{\theta}$.
(f) Compare the mean squared errors for the two types of estimators in parts (c) and (d). *Hint : some of the required calculations have been already carried out in the course.*

Assignment 7. Let $X_1, \ldots, X_n$ be a sample from a Poisson(λ) distribution.
(a) Write the minimal sufficient statistics T and call $\overline{T} = T/n$. Use theorem in Slide 100 to compute the mean and the variance of both T and $\overline{T}$.
(b) Use the theorem on Slide 114 to find the approximate sampling distribution for T.

Assignment 8. Last week we have found a sufficient statistics for some member of the exponential family. This week we focus on minimally sufficient.

(i) Prove that $T(z) = \sum_{i=1}^n z_i$ is a minimal sufficient statistics for the Binomial distribution.
(i) Prove that $T(y) = \sum_{i=1}^n y_i$ is a minimal sufficient statistics for the Poisson distribution.
(iii) Prove that $T_1(y) = \sum_{i=1}^n y_i$ is a minimal sufficient statistics for the mean of a Normal $\mathcal{N}(\mu, \sigma^2)$ distribution and $T = (T_1, T_2)$ with $T_2(y) = \sum_{i=1}^n y_i^2$ is a minimum sufficient statistics for σ (and thus for both parameters).